November 5, 2009 **Name** ______________________________

The problems count as marked. The total number of points available is 155. Throughout this test, **show your work.**

1. (25 points) Let $f(x) = 3x^4 + 4x^3 - 72x^2 + 2$.

   (a) Find the intervals over which f is increasing.

   Solution: First note that $f'(x) = 12x^3 + 12x^2 - 144x$, which factors into $12x(x+4)(x-3)$, so we can build the sign chart for f' using the branch points $0, -4$ and 3. Doing this yields the result that f' is positive over each of the intervals $(-4, 0)$ and $(3, \infty)$. Therefore f is increasing over these two intervals.

   (b) Find $f(0)$ and use this together with your answer to part (a) to sketch the graph of f.

   Solution: First note that $f(0) = 2$. Since $f'(x) = 12x^3 + 12x^2 - 144x$, it follows that $f'(0) = 0$.

   (c) Find $f'(0)$ and use this with the information in part (b) to find an equation for the line tangent to the graph of f at the point $(0, f(0))$.

   Solution: Since $f(0) = 2$, the tangent line is just $y - 2 = 0(x - 0) = 0$ or $y = 2$.

2. (20 points) Suppose the function g has been differentiated twice to get $g''(x) = (x-4)(x+1)(2x+9)$.

   (a) Construct the sign chart for g''.

   Solution: Note that g'' is positive on $(-9/2, -1)$ and on $(4, \infty)$.

   (b) Find the intervals over which the function g is concave upwards.

   Solution: Same as above.

3. (20 points) Let $f(x) = (x^2-4)^{2/3}$. Find $f'(x)$. Find all the critical points and identify each one as the location of a relative Maximum, a relative minimum, or neither (an imposter).

   Solution: $f'(x) = 2(x^2-4)^{-1/3} \cdot 2x/3$, which has $x = 0$ as a zero and $x = \pm 2$ as points of undefinedness. Therefore $x = 0$ is a *stationary* point and $x = \pm 2$ are *singular* critical points.

4. (20 points) Compute the following derivatives.

   (a) Let $r(x) = (x^2 - x) \cdot e^{2x-3}$. Find $\frac{d}{dx}r(x)$. Recall that $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$.

   Solution: We differentiate it using the power rule and chain rule: $r'(x) = (2x-1)e^{2x-3} + 2e^{2x-3}(x^2-x)$.

   (b) Use the fact that $e^x \neq 0$ for all x to find the critical points of the function r in part (a).

   Solution: After factoring, $r'(x) = e^{2x-3}[2x-1+2(x^2-x)] = e^{2x-3}[-1+2x^2]$, which is zero precisely when $x = \pm\sqrt{1/2}$.

   (c) Let $k(x) = \sqrt{x^3 - 6x^2 + 5x - x^{-1}}$. What is $k'(x)$?

   Solution: Use the chain rule to get $k'(x) = (x^3 - 6x^2 + 5x - x^{-1})^{-1/2} \cdot (3x^2 - 12x + 5 + x^{-2})/2$.

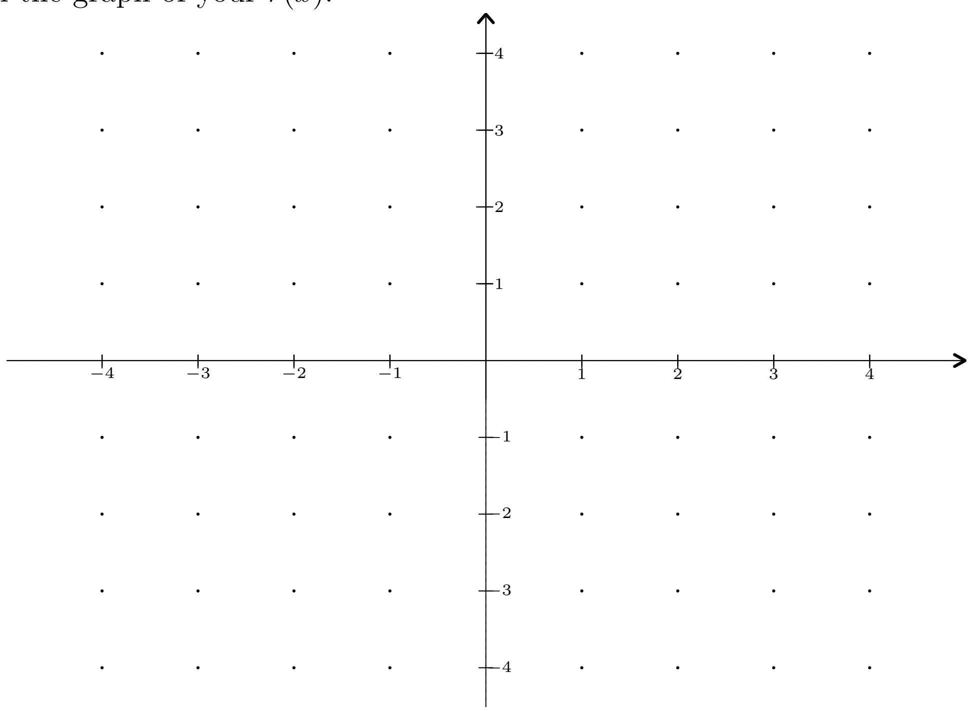
   (d) Let $g(x) = \frac{2x^3+1}{3x-2}$. Find $g'(x)$.

   Solution: By the quotient rule, $\frac{d}{dx}\frac{2x^3+1}{3x-2} = \frac{6x^2(3x-2)-3(2x^3+1)}{(3x-2)^2}$. This expression simplifies to one with a numerator $12x^3 - 12x^2 - 3$.

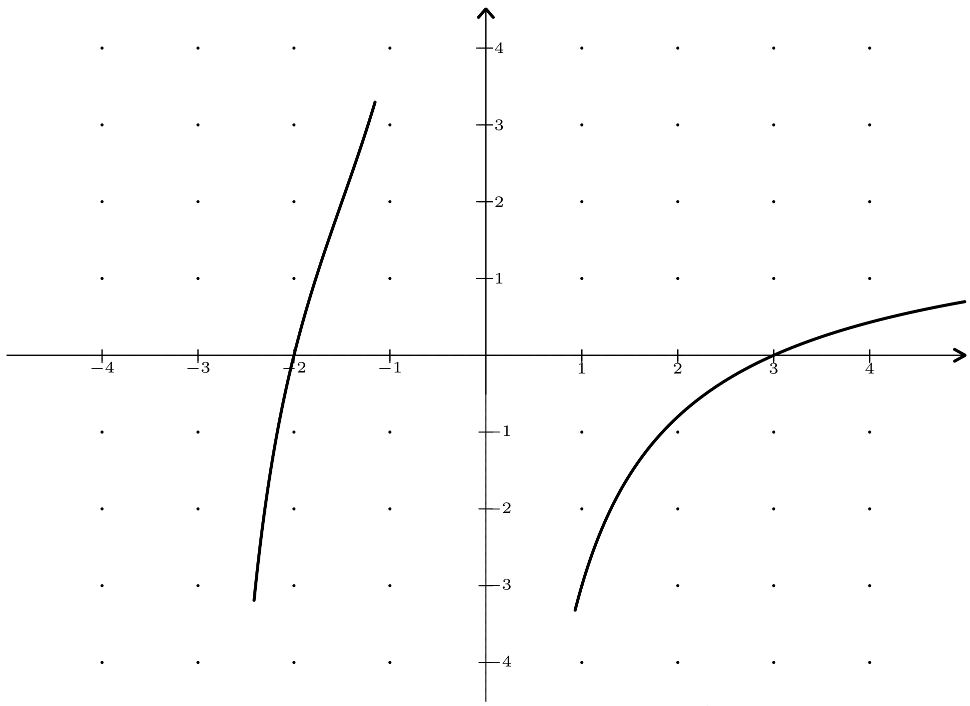
5. (20 points) Find a rational function $r(x)$ that has all the following properties:

   - It has exactly two zeros, $x = -2$ and $x = 3$.
   - It has two vertical asymptotes, $x = 0$ and $x = -3$.
   - It has $y = 2$ as a horizontal asymptote.

   (a) Sketch the graph of your $r(x)$.



Solution:



Sadly, you cannot see the little curly part in the upper left corner.

(b) Find a symbolic representation of r.

Solution: There are a few ways to do this. The easiest is to make the numerator $2(x+2)(x-3)$ and the denominator $x(x+3)$. The graph is shown above. So

$$r(x) = \frac{2(x+2)(x-3)}{x(x+3)}.$$

6. (20 points) A baseball team plays in he stadium that holds 60000 spectators. With the ticket price at 12 dollars the average attendance has been 25000. When the price dropped to 10 dollars, the average attendance rose to 40000.

   (a) Find the demand function $p(x)$, where x is the number of the spectators and $p(x)$ is measured in dollars, assuming it is linear. In other words, if the relationship between the price and number of tickets sold is linear, find the price when x tickets are sold.

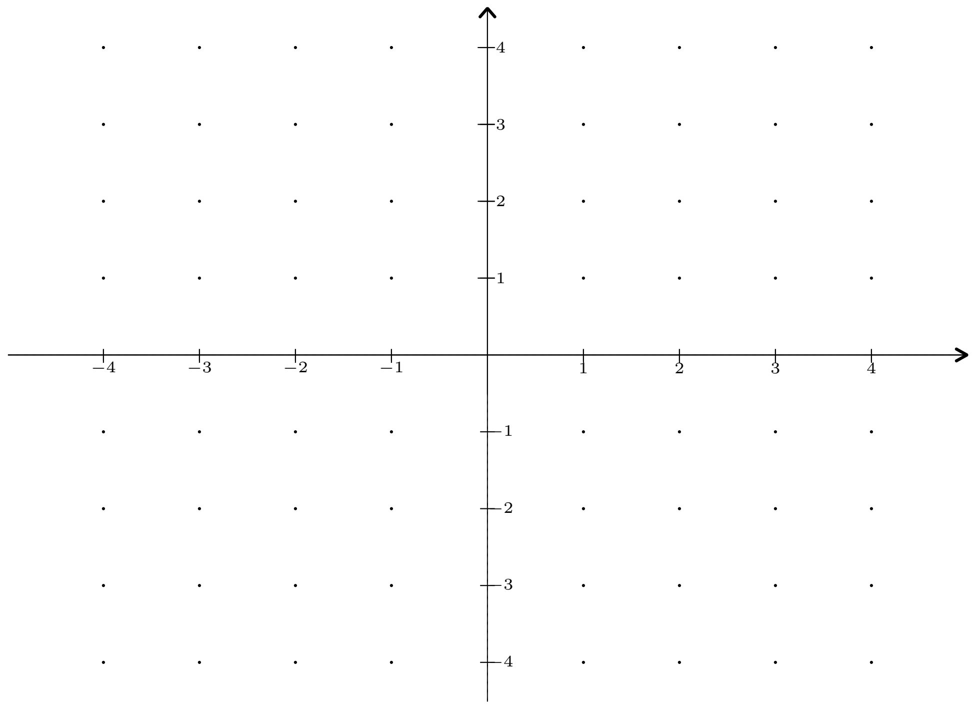
   Solution: We need to find the linear demand function, given that $(25000, 12)$ and $(40000, 10)$ are on the graph. To simplify, we measure the attendance in thousands, so the two points are $(25, 12)$ and $(40, 10)$. Thus the slope is $m = \frac{12-10}{25-40} = -\frac{2}{15}$. Using the point-slope form of a line, we have $p(x) - 10 = -2/15(x - 40)$. Simplifying yields $p(x) = (-2x + 230)/15$.

   (b) How should the ticket price be set to maximize revenue?

   Solution: Now the revenue function $R(x)$ is the product of number of tickets sold and the price per ticket. Thus $R(x) = xp(x) = x(-2x + 230)/15$. $R'(x) = (-4x+230)/15$, which has a zero at $x = 230/4 = 57.50$. What this says is that the optimum attendance is $57.50 \cdot 1000 = 57500$ and that corresponds to a ticket price of $23/3$ dollars.

7. (10 points) Sketch the graph of the function

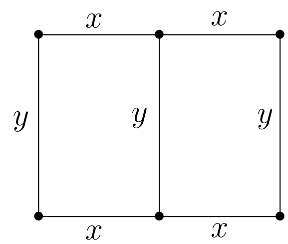
$$f(x) = \frac{|\, x-1 \,|}{x-1} + \frac{|\, x+3 \,|}{x+3}.$$



Solution: We must eliminate precisely the values $x = 1$ and $x = -3$, so the set is $(-\infty, -3) \cup (-3, 1) \cup (1, \infty)$.

8. (20 points) A rancher wants to fence in an area of 10 square miles in a rectangular field and then divide it in half with a fence down the middle parallel to one side. What is the shortest length of fence that the rancher can use?

Solution: About 16 miles of fencing is needed. See the diagram below. Note that the total amount of fencing needed, based on the labeling of the figure is $3y + 4x$ and the area fenced in is $A = 10 = 2xy$. Solve the last relation for y to get $y = 5/x$. Now the amount of fencing f can be written in terms of x as follows: $f(x) = 3(5/x) + 4x, 0 < x$. Find the critical points of f by first noting that $f'(x) = 15(-1)x^{-2} + 4$. Then solve $f'(x) = 0$ to get $x = \sqrt{15}/2$. The sign chart for f' shows that f has a minimum at $\sqrt{15}/2$. The rancher needs $f(\sqrt{15}/2) = 4\sqrt{15} \approx 15.49$ miles of fencing.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