March 17, 2010 **Name** ______________________________

The problems count as marked. The total number of points available is 164. Throughout this test, **show your work.**

1. (24 points) Demonstrate your understanding of the product, quotient and chain rules by differentiating each of the given functions. No need to simplify. You must show your work.

   (a) Let $F(x) = (x^2 - 3x + 1)(x^3 - 2x + 5)$

   Solution: Note that $F'(x) = (2x-3)(x^3-2x+5)+(3x^2-2)(x^2-3x+1)$.

   (b) $G(x) = \dfrac{2x^4 - 3x + 1}{x^2 - x + 3}$

   Solution: By the quotient rule, $G'(x) = \frac{(8x^3-3)(x^2-x+3)-(2x-1)(2x^4-3x+1)}{(x^2-x+3)^2}$.

   (c) $K(x) = (x^2 - 3)^{17}$

   Solution: By the chain rule, $K'(x) = 17(x^2 - 3)^{16} \cdot 2x = 34x(x^2 - 3)^{16}$.

   (d) $H(x) = \sqrt{(3x + 1)^4 - 7}$.

   Solution: By the chain rule, $H'(x) = \frac{1}{2}((3x+1)^4-7)^{-1/2} \cdot 4(3x+1)^3 \cdot 3 = \frac{6(3x+1)^3}{\sqrt{(3x+1)^4-7}}$.

2. (10 points) The line tangent to the graph of $g(x)$ at the point $(4, 7)$ has a y-intercept of 9. What is $g'(4)$?

   Solution: The line has slope $(9 - 7)/(0 - 4) = -1/2$.

3. (10 points) Find a point on the graph of $h(x) = x^3 - 6x^2 + 9x$ where the tangent line is horizontal. There are two such points on the graph of $h(x)$.

   Solution: Since $h'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)$, it follows that h has a horizontal tangent line for $x = 1$ and $x = 3$. The points on the graph are $(1, 4)$ and $(3, 0)$.

4. (25 points) A stone is thrown upwards from the top of a 200 foot high building in such a way that its height is given by

$$s(t) = -16t^2 + 128t + 200$$

feet, where t is measured in seconds.

(a) Find the height of the stone for each value of t listed:

i. $t = 0$
Solution: $s(0) = 200$ feet.

ii. $t = 1$
Solution: $s(1) = -16 + 128 + 200 = 312$ feet.

iii. $t = 2$
Solution: $s(2) = -64 + 256 + 200 = 392$ feet.

iv. $t = 2.1$
Solution: $s(2.1) = 398.24$ feet.

v. $t = 2.01$
Solution: $s(2.01) = -64.6416 + 128(2.01) + 200 = 392.63$ feet.

(b) How far did the stone travel during the first two seconds of its flight?
Solution: $s(2) - s(0) = 392 - 200 = 192$ feet.

(c) What was the average speed of the stone during those first two seconds?
Solution: $(s(2) - s(0))/2 = 192/2 = 96$ feet per second.

(d) How far did the stone travel during the time interval $[2, 2.1]$ and what was its average velocity during that time?
Solution: $(s(2.1)-s(2))/(2.1-2) = (398.24-392)/(2.1-2) = 6.24/0.1 = 62.4$ feet per second.

(e) How far did the stone travel during the time interval $[2, 2.01]$ and what was its average velocity during that time?
Solution: $(s(2.01) - s(2))/(2.01 - 2) = (392.63 - 392)/(2.011 - 2) = 0.63/0.01 = 63.0$ feet per second.

(f) What is $s'(t)$? What is the relation between $s'(2)$ and the numbers you found in parts (d) and (e).
Solution: $s'(t) = -32t + 128$, so $s'(2) = v(2) = -64 + 128 = 64$ feet per second. The relationship between the 62.4 and the 63 on one hand and $s'(2)$ on the other is that the first two are estimates of the last. In fact $s'(2) = \lim_{t \to 2^+} \frac{s(t)-s(2)}{t-2}$.

(g) What is the maximum height attained by the stone?
Solution: To see where the stone stops, solve $v(t) = -32t + 128 = 0$ to get $t = 4$, and then find $s(4) = -16(16) + 128(4) + 200 = 456$ feet.

5. (35 points) Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	3	2
1	4	6	2	5
2	6	4	3	4
3	1	2	5	3
4	3	5	2	6
5	5	3	4	1
6	0	3	1	4

(a) $Q(x) = f(x)/g(x)$. Find $Q'(5)$.

Solution: $Q'(x) = (f'(x)g(x) - g'(x)f(x))/(g(x))^2$. Therefore, $Q'(5) = (f'(5)g(5) - g'(5)f(5))/(g(5))^2 = \frac{3\cdot 4 - 1\cdot 5}{4^2} = \frac{7}{16}$.

(b) Let $H(x) = f(x) \cdot (g(x) + 1)$. Compute $H'(4)$.

Solution: By the product and chain rules, $H'(x) = f'(x) \cdot (g(x) + 1) + g'(x) \cdot f(x)$. Therefore, $H'(4) = f'(4) \cdot (g(4)+1) + g'(4) \cdot f(4) = 5\cdot 3 + 6\cdot 3 = 33$.

(c) Let $W(x) = f(g(x) + 1)$. Compute $W'(5)$.

Solution: Again by the chain rule, $W'(x) = f'(g(x) + 1) \cdot g'(x)$, so $W'(5) = f'(g(5) + 1) \cdot (g'(5)) = 3 \cdot 1 = 3$.

(d) Let $L(x) = g(\frac{1}{x} + 1)$. Compute $L'(1)$.

Solution: By the chain rule, $L'(x) = g'(\frac{1}{x} + 1) \cdot (-x^{-2})$, so $L'(2) = g'(\frac{1}{1} + 1) \cdot (-1^{-2}) = g'(2) \cdot (-1) = -4$.

(e) Let $U(x) = \sqrt{g(2x)}$. Compute $U'(3)$.

Solution: By the chain rule, $U'(x) = \frac{1}{2} g(2x)^{-1/2} \cdot g'(2x) \cdot 2$, so $U'(3) = \frac{1}{2} g(6)^{-1/2} \cdot g'(6) \cdot 2 = \frac{1}{2} \cdot 1 \cdot 4 \cdot 2 = 4$.

(f) Let $Z(x) = g(2x - f(x))$. Compute $Z'(4)$.

Solution: Again by the chain rule and the product rule, $Z'(x) = g'(2x - f(x)) \cdot (2 - f'(x))$ so $Z'(4) = g'(8 - f(4)) \cdot (2 - f'(4)) = g'(5)(2 - 5) = 1(-3) = -3$.

6. (30 points) The cost of producing widgets is given by $C(x) = 10000 + 50x - 0.003x^2, \quad 0 \leq x \leq 1000$. The relationship between price and demand for widgets is given by $p = f(x) = -0.04x + 300, \quad 0 \leq x \leq 7000$, where p is the price in dollars.

   (a) Find the average cost function $\overline{C}(x)$.
   Solution: $\overline{C}(x) = 10000/x + 50 - 0.003x$.

   (b) Find the (incremental) cost of producing the 500^{th} widget.
   Solution: $C(500) - C(499) = 10000 + 50 \cdot 500 - 0.003 \cdot 500^2 - (10000 + 50 \cdot 499 - 0.003 \cdot 499^2) = 50 - 0.003(999) = 47.003$.

   (c) Find the marginal cost function $C'(x)$.
   Solution: $C'(x) = 50 - 0.006x$.

   (d) What is $C'(500)$?
   Solution: $C'(500) = 50 - 0.006(500) = 50 - 3 = 47$.

   (e) Find the marginal average cost function $\overline{C}'(x)$.
   Solution: $\overline{C}'(x) = -10000/x^2 - 0.003$.

   (f) Find the revenue function $R(x)$.
   Solution: $R(x) = xp = xf(x) = x(-0.04x + 300) = -.04x^2 + 300x$.

   (g) Find the marginal revenue function $R'(x)$.
   Solution: $R'(x) = -0.08x + 300$.

   (h) Find the profit function $P(x)$.
   Solution: $P(x) = R(x) - C(x) = 250x - 0.037x^2 - 10000$.

   (i) Find the marginal profit function $P'(x)$.
   Solution: $P'(x) = 250 - 0.074x$.

   (j) Find a value of x where the profit function $P(x)$ has a horizontal tangent line.
   Solution: Solve $P'(x) = 0$ for x to get $x = 250 \div 0.074 \approx 3378$.

7. (30 points) Let $g(x) = (x^2 - 4)^2(2x+1)^2$. Using the chain and product rules, we can differentiate $g(x)$ to get to find

$$g'(x) = 2(x^2-4)(2x)(2x+1)^2 + 2(2x+1) \cdot 2(x^2-4)^2.$$

(a) Find all the x-intercepts (the zeros) of $g'(x)$. That is, find the critical points of g.

Solution: Factor out $4(x^2-4)(2x+1)$ from both terms to get $g'(x) = 4(x^2-4)(2x+1)[x(2x+1)+x^2-4] = 4(x^2-4)(2x+1)[3x^2+x-4] = 4(x-2)(x+2)(2x+1)(3x+4)(x-1)$, so there are five distinct critical points, $-2, -4/3, -1/2, 1$ and 2.

(b) Build the sigh chart for $g'(x)$.

Solution: $g'(x)$ is positive on the intervals $(-2,-4/3), (-1/2, 1)$ and $(2, \infty)$.

(c) Use the sign chart for $g'(x)$ to classify each critical point of g found in part (a) as the location of (i) a local minimum, (ii) a local maximum, or (iii) an imposter.

Solution: Since g is increasing over the intervals where g' is positive, it follows that $g(x)$ has a minimum at each of the points $-2, -1/2$, and 2, and maxima at the other two points $-4/3$ and 1.