November 4, 2010 **Name** ______________________________

The problems count as marked. The total number of points available is 145. Throughout this test, **show your work.**

1. (10 points) Suppose f and g are functions for which both $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. Which of the following is true? Circle your answer.

   (A) $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$ **(B)** $\lim_{x\to a} \frac{f(x)}{g(x)}$ does not exist **(C)** $\lim_{x\to a} \frac{f(x)}{g(x)} = \infty$

   (D) $\lim_{x\to a} \frac{f(x)}{g(x)}$ can be any real number **(E)** $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$

   Solution: We know this limit can be any real number. For example, let r be any number and let $f(x) = r(x-a)$ and $g(x) = x - a$. Then

   $$\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} r = r.$$

2. (20 points) Let $f(x) = \sqrt{x^4 - 3x + 11}$.

   (a) Compute $f'(x)$
   Solution: $f'(x) = \frac{1}{2}(x^4 - 3x + 11)^{-1/2}(4x^3 - 3) = \frac{4x^3-3}{2\sqrt{x^4-3x+11}}$.

   (b) What is $f'(1)$?
   Solution: $f'(1) = \frac{4-3}{2\sqrt{1-3+11}} = 1/6$

   (c) Use the information in (b) to find an equation for the line tangent to the graph of f at the point $(1, f(1))$.
   Solution: Since $f(1) = 3$, using the point-slope form leads to $y - 3 = f'(1)(x-1) = (x-1)/6$, so $y = x/6 + 17/6$.

3. (20 points) Use calculus to find all relative max and min and also all asymptotes of the function $g(x) = 4x + 16/x$.

   Solution: First, note that $g'(x) = 4 - 16/x^2$, which has two zeros, $x = 2$ and $x = -2$. The sign chart for $g'(x)$ reveals that g has a maximum at $x = -2$ and a minimum at $x = 2$. To check the asymptotes, write the function in standard rational function form, $g(x) = \frac{4x^2+16}{x}$. Using the Asymptote Theorem, we see that $x = 0$ is a vertical asymptote, and since the degree of the numerator (2) is larger than the degree of the denominator (1), there is no horizontal asymptote.

   Then, use calculus to discuss the concavity of $g(x)$.

   Solution: The sign chart for $g''(x) = 32/x^3$ is quite simple: negative on $(-\infty, 0)$, and positive of $(0, \infty)$. So g is concave downward over the first and upward over the second.

4. (20 points) Let
$$H(x) = (2x + 1)^2(3x - 4)^2.$$

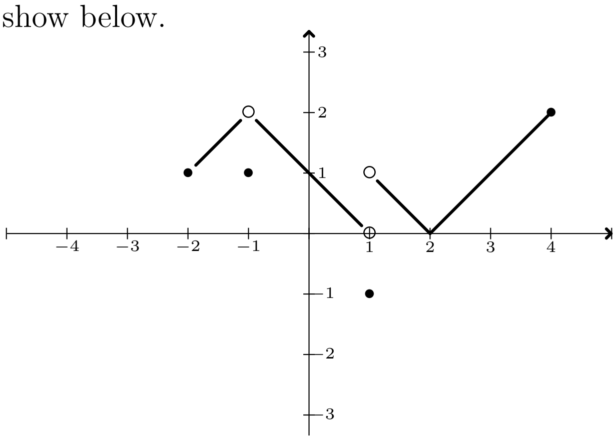
   (a) For what values of x is the line tangent to the graph of $H(x)$ parallel to the line $y = 7$?

   Solution: Since $f'(x) = 2(2x+1)\cdot 2\cdot (3x-4)^2+2(3x-4)\cdot 3\cdot (2x+1)^2$, we seek those values of x such that $2(2x+1)(3x-4)[2(3x-4)+3(2x+1)] = 0$. Factoring and simplifying yields $x = -1/2, x = 4/3$ and $6x-8+6x+3 = 0$, so the three values of are $x = -1/2, x = 4/3$ and $x = 5/12$.

   (b) Find the intervals over which $H(x)$ is increasing.

   Solution: Building the sign chart for $H'(x)$, we see that H is increasing precisely over the intervals $(-1/2, 5/12)$ and $(4/3, \infty)$.

5. (20 points) Find the symbolic representation of the function G whose graph is show below.



As a hint, the function needs six clauses, as shown.

$$G(x) = \begin{cases} & \text{if } -2 \le x < -1 \\ & \text{if } x = -1 \\ & \text{if } -1 < x < 1 \\ & \text{if } x = 1 \\ & \text{if } 1 < x < 2 \\ & \text{if } 2 \le x \le 4 \end{cases}$$

Solution:

$$G(x) = \begin{cases} x+3 & \text{if } -2 \le x < -1 \\ 1 & \text{if } x = -1 \\ 1-x & \text{if } -1 < x < 1 \\ -1 & \text{if } x = 1 \\ 2-x & \text{if } 1 < x < 2 \\ x-2 & \text{if } 2 \le x \le 4 \end{cases}$$

6. (20 points) Use calculus to find the point $P = (u, v)$ on the line $2x + 3y = 7$ that is closest to the origin $(0, 0)$. Then use geometry to check your answer. Is the slope of the line $y = (v/u)x$ right? Write a complete sentence about your reasoning.

Solution: For each x, the point on the line above x is $(x, -2x/3+7/3)$, so the function $D(x)$ that we are minimizing is $D(x) = \sqrt{(x-0)^2 + (-2x/3 + 7/3 - 0)^2}$, and the derivative $D'(x)$ is given by

$$D'(x) = \frac{1}{2}\left(x^2 + (-2x/3 + 7/3))^2\right)^{-1/2} \cdot (2x + 2(-2x/3 + 7/3)(-2/3)).$$

Setting this equal to zero, and noting that we need only look at the part after the dot, we find $2x + 8x/9 - 28/9 = 0$, which leads to $26x/9 = 28/9$, and then to $x = 14/13$. The corresponding y is $y = -2(14/13)/3 + 7/3 = 63/39 = 21/13$. This means that the slope of the line from the origin to (x, y) is $21/13 \div 14/13 = 3/2$, just as the geometry predicts. The slope of the line give is $-2/3$, so the two are perpendicular as we know they have to be.

7. (35 points) Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	3	2
1	4	6	2	5
2	6	4	3	4
3	1	2	5	3
4	3	5	2	6
5	5	3	4	1
6	0	3	2	4

(a) Let $L(x) = (f(x) + g(x))^2$. Compute $L'(2)$.

Solution: $L'(x) = 2(f(x) + g(x))(f'(x) + g'(x))$, so $L'(2) = 2(f(2) + g(2))(f'(2) + g'(2)) = 2(6 + 3)(4 + 4) = 144$.

(b) Let $U(x) = f \circ f \circ f(x)$. Compute $U'(1)$.

Solution: By the chain rule, $U'(x) = f'(f \circ f(x)) \cdot f'((f(x)) \cdot f'(x)$, so $U'(1) = f'(f \circ f(1)) \cdot f'((f(1)) \cdot f'(1) = f'(f((f(1)) \cdot f'(f(1)) \cdot f'(1) = f'(3) \cdot f'(4) \cdot f'(1) = 2 \cdot 5 \cdot 6 = 60$.

(c) Let $K(x) = g(x) + f(x^2)$. Compute $K'(2)$

Solution: $K'(x) = g'(x) + f'(x^2)2x$, so $K'(2) = g'(2) + f'(2^2)2 \cdot 2 = 4 + 5 \cdot 4 = 24$.

(d) Let $Z(x) = f(x) \div g(x)$. Compute $Z'(3)$.

Solution: By the quotient rule, $Z'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$ so $Z'(3) = \frac{f'(3)g(3) - g'(3)f(3)}{(g(3))^2} = \frac{2 \cdot 5 - 3 \cdot 1}{5^2} = 7/25$.

(e) Let $Q(x) = g(3x) \cdot f(2x)$. Compute $Q'(2)$.

Solution: Again by the product rule and chain rule, $Q'(x) = 3g'(3x) \cdot f(2x) + 2f'(2x) \cdot g(3x)$ so $Q'(2) = 3g'(6) \cdot f(4) + 2f'(4) \cdot g(6) = 3 \cdot 4 \cdot 3 + 2 \cdot 5 \cdot 2 = 36 + 20 = 56$.