March 2, 2011 **Name** ______________________________

The problems count as marked. The total number of points available is 128. Throughout this test, **show your work.**

1. (12 points) Let $f(x) = x^2 - 2x - 3$.

   (a) Compute $f'(x)$
   Solution: $f'(x) = 2x - 2$

   (b) What is $f'(2)$?
   Solution: $f'(2) = 2$.

   (c) Use the information in (b) to find an equation for the line tangent to the graph of f at the point $(2, f(2))$.
   Solution: Since $f(2) = 2^2 - 2 \cdot 2 - 3 = -3$, using the point-slope form leads to $y + 3 = f'(2)(x - 2) = 2(x - 2)$, so $y = 2x - 7$.

2. (12 points) Consider the function f defined by:

$$f(x) = \begin{cases} x + x^3 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 2x^{1/2} & \text{if } x > 1 \end{cases}$$

   (a) Find an equation for the line tangent to the graph of f at the point $(4, 4)$.
   Solution: To find $f'(4)$ first note that when x is near 8, $f(x) = 2x^{1/2}$ so $f'(x) = 2\frac{1}{2}x^{-1/2}$. Thus, $f'(4) = 2\frac{1}{2}4^{-1/2} = 2\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$. So the line in question is $y - 4 = \frac{1}{2}(x - 4)$, which, in slope-intercept form is $y = x/2 + 2$.

   (b) Find an equation for the line tangent to the graph of f at the point $(-2, -10)$.
   Solution: Note that $f'(x) = 1 + 3x^2$ when $x < 1$. So we have $f'(-2) = 1 + 3(-2)^2 = 13$. So the tangent line has slope 13. Thus, $y + 10 = 13(x + 2)$, which can also be written $y = 13x + 16$.

3. (12 points) The function f satisfies $f(2) = 5$ and its graph has a tangent line L at the point $(2, 5)$. The line L has a y-intercept of 4. What is $f'(2)$? Note, a correct answer without supporting work is worth only 1 point.

   Solution: The tangent line goes through the two points $(2, 5)$ and $(0, 4)$, so its slope is $\frac{5-4}{2-0} = \frac{1}{2}$, so $f'(2) = 1/2$.

4. (15 points) If a ball is thrown vertically upward from the roof of 212 foot building with a velocity of 48 ft/sec, its height after t seconds is $s(t) = 212 + 48t - 16t^2$.

   (a) What is the height the ball at time $t = 0$?

   Solution: $s(0) = 212$.

   (b) What is the velocity of the ball at the time it reaches its maximum height?

   Solution: $s'(t) = v(t) = 0$ when the ball reaches its max height.

   (c) At what time is the velocity zero?

   Solution: Solve $48 - 32t = 0$ to get $t = 3/2$.

   (d) What is the maximum height the ball reaches?

   Solution: Solve $s'(t) = 48 - 32t = 0$ to get $t = 3/2$ when the ball reaches its zenith. Thus, the max height is $s(3/2) = 212 + 48(3/2) - 16(3/2)^2 = 248$.

   (e) What is the velocity of the ball when it hits the ground (height 0)?

   Solution: Solve $s(t) = 0$ using the quadratic formula to get $t = \frac{3 \pm \sqrt{9+53}}{2} = \frac{3 \pm \sqrt{62}}{2} \approx 5.44$, since the larger is the only reasonable answer. Find $s'(5.44) \approx -125.98$ feet/sec.

5. (10 points) Find a point on the graph of f at which the slope of the tangent line is 3, where
$$f(x) = x - \frac{1}{x}.$$
Are there any other points where the slope is 3?

Solution: $f'(x) = 1+\frac{1}{x^2}$, which we set equal to 3 to get $1/x^2 = 2$, or $x = \sqrt{2}/2$. The other solution also works, $x = -\sqrt{2}/2$.

6. (10 points) The cost of producing x units of stuffed alligator toys is $C(x) = -0.003x^2 + 6x + 6000$ for $0 \leq x \leq 1000$.

   (a) Find the marginal cost at the production level of 500 units.

   Solution: $C'(x) = \frac{d}{dx} - 0.003x^2 + 6x + 6000 = -0.006x + 6$ so $C'(500) = -3 + 6 = 3$.

   (b) Find the (incremental) cost of producing the 501^{st} toy.

   Solution: $C(501) - C(500) = -0.003(501^2 - 500^2) + 6(501 - 500) + 6000 - 6000 = -0.003(1001) + 6 = 2.997$.

7. (32 points) Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	3	2
1	4	6	2	5
2	6	4	3	4
3	1	2	5	3
4	3	5	2	6
5	5	3	4	1
6	0	3	2	4

(a) Let $L(x) = (f(x) + g(x))^3$. Compute $L'(2)$.

Solution: $L'(x) = 3(f(x) + g(x))^2(f'(x) + g'(x))$, so $L'(2) = 3(f(2) + g(2))^2(f'(2) + g'(2)) = 3(6 + 3)^2(4 + 4) = 1944$.

(b) Let $U(x) = g \circ g \circ g(x)$. Compute $U'(1)$.

Solution: By the chain rule, $U'(x) = g'(g \circ g(x)) \cdot g'(g(x)) \cdot g'(x)$, so $U'(1) = g'(g \circ g(1)) \cdot g'(g(1)) \cdot g'(1) = g'(g(g(1))) \cdot g'(g(1)) \cdot g'(1) = g'(3) \cdot g'(2) \cdot g'(1) = 3 \cdot 4 \cdot 5 = 60$.

(c) Let $K(x) = \frac{g(x)+f(x)}{g(x)f(x)}$. Compute $K'(2)$

Solution: $K'(x) = \frac{(g'(x)+f'(x))(g(x)f(x))-(f'(x)g(x)+g'(x)f(x))(f(x)+g(x))}{(f(x)g(x))^2}$. So $K'(2) = \frac{(g'(2)+f'(2))(g(2)f(2))-(f'(2)g(2)+g'(2)f(2))(f(2)+g(2))}{(f(2)g(2))^2} = \frac{36}{81} = \frac{4}{9}$.

(d) Let $Z(x) = x^2 - \frac{f(x)}{x}$. Compute $Z'(3)$.

Solution: By the quotient rule, $Z'(x) = 2x - \frac{f'(x)(x)-1f(x)}{x^2}$ so $Z'(3) = 6 - \frac{f'(3)(3)-f(3)}{9} = 6 - \frac{2 \cdot 3 - 1}{9} = 49/9$.

8. (25 points) Let $H(x) = (x^2 - 1)^2(5x + 7) + (x^2 - 1)(5x + 7)^2$.

   (a) Build the sign chart for $H(x)$.

   Solution: Factor out the common terms to get $H(x) = (x^2 - 1)(5x + 7)[x^2 - 1 + 5x + 7]$. The zeros of $x^2 - 1$ are $x = \pm 1$ and the zero of $5x + 7$ is $x = -7/5$. The quadratic $x^2 + 5x + 6$ factors into $(x + 3)(x + 2)$, so its two zeros are $x = -3$ and $x = -2$.

   (b) Use the information from (a) to find the domain of the function $G(x) = \sqrt{(x^2 - 1)^2(5x + 7) + (x^2 - 1)(5x + 7)^2}$. Express your answer in interval notation.

   Solution: Looking at the sign chart above, we have $[-3, -2] \cup [-\frac{7}{5}, -1] \cup [1, \infty)$.