October 27, 2011 **Name** ______________________________

The problems count as marked. The total number of points available is 148. Throughout this test, **show your work.**

1. (9 points) Let $f(x) = x^4 - 1/x - 3$.

   (a) Compute $f'(x)$

   Solution: $f'(x) = 4x^3 + x^{-2}$

   (b) What is $f'(1)$?

   Solution: $f'(1) = 4 \cdot 1^3 + 1 = 5$.

   (c) Use the information in (b) to find an equation for the line tangent to the graph of f at the point $(1, f(1))$.

   Solution: Since $f(1) = 1^4 - 1/1 - 3 = -3$, using the point-slope form leads to $y + 3 = f'(1)(x - 1) = 5(x - 1)$, so $y = 5x - 8$.

2. (12 points) Consider the function f defined by:

$$f(x) = \begin{cases} x + x^3 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 2x^{3/2} & \text{if } x > 1 \end{cases}$$

   (a) Is f continuous at $x = 1$?

   Solution: Yes, the limits from the left and right are both 2, and the value of f at 1 is 2.

   (b) What is the slope of the line tangent to the graph of f at the point $(4, 16)$?

   Solution: To find $f'(4)$ first note that when x is near 8, $f(x) = 2x^{3/2}$ so $f'(x) = 2 \cdot \frac{3}{2} \cdot x^{1/2}$. Thus, $f'(4) = 2 \cdot \frac{3}{2} \cdot 4^{1/2} = 3 \cdot 2 = 6$.

   (c) Find $f'(-3)$

   Solution: To find $f'(-3)$, we must differentiate the part of f defined for $x < 1$. In this area, $f'(x) = 1 + 3x^2$, so $f'(-3) = 1 + 3(-3)^2 = 28$.

3. (10 points) The cost of producing x units of stuffed alligator toys is $C(x) = -0.003x^2 + 6x + 6000$ for $0 \leq x \leq 1000$.

   (a) Find the marginal cost at the production level of 1000 units.
   Solution: $C'(x) = \frac{d}{dx} - 0.003x^2 + 6x + 6000 = -0.006x + 6$ so $C'(1000) = -6 + 6 = 0$.

   (b) Find the (incremental) cost of producing the 1000^{th} toy.
   Solution: $C(1000) - C(999) = -0.003(1000 - 999)^2 + 6(1000 - 999) + 6000 - 6000 = -0.003(1999) + 6 = 0.003$.

4. (15 points) Consider the function $f(x) = x^3 - 6x$ defined on the interval $-2 \leq x \leq 3$.

   (a) Find the critical points of f.
   Solution: Since $f'(x) = 3x^2 - 6$, the critical points at $x = \pm\sqrt{2}$.

   (b) Find the absolute minimum of f and the x-value where it occurs.
   Solution: Evaluate f at the critical points and the endpoints to find $f(-\sqrt{2}) = 4\sqrt{2}$, $f(\sqrt{2}) = -4\sqrt{2}, f(-2) = 4$, and $f(3) = 9$. So the absolute minimum of f is $-4\sqrt{2}$ and it occurs at $x = \sqrt{2}$.

   (c) Find the absolute maximum of f and the x-value where it occurs.
   Solution: The maximum value of f is 9 and it occurs at $x = 3$

5. (30 points) Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	6	2
1	4	6	2	5
2	3	4	2	3
3	1	2	5	3
4	3	5	2	5
5	5	3	4	1
6	0	3	2	4

(a) Let $L(x) = f(x) \cdot g(x)$. Compute $L'(5)$.
Solution: $L'(x) = f'(x)g(x)+f(x)g'(x)$, so $L'(5) = f'(5)g(5)+f(5)g'(5) = 3 \cdot 4 + 5 \cdot 1 = 17$.

(b) Let $U(x) = f \circ f(x)$. Compute $U'(4)$.
Solution: $U'(x) = f'(f(x))f'(x)$ so $U'(4) = f'(f(4))f'(4) = 2 \cdot 5 = 10$.

(c) Let $K(x) = (g(x) + f(x))^3$. Compute $K(2)$.
Solution: $K(2) = (g(2) + f(2))^3 = (2 + 3)^3 = 125$.

(d) Again, $K(x) = (g(x) + f(x))^3$. Compute $K'(2)$.
Solution: $K'(x) = 3(g(x) + f(x))^2 \cdot (g'(x) + f'(x))$, so $K'(2) = 3(2 + 3)^2(3 + 4) = 525$.

(e) Let $V(x) = f(x^2) \div g(x)$. Compute $V'(2)$.
Solution: By the quotient rule, $V'(x) = [f'(x^2) \cdot 2x \cdot g(x) - g'(x)f(x^2)] \div (g(x))^2$, so $V'(2) = [f'(4) \cdot 2 \cdot 2g(2) - g'(2) \cdot f(4)] \div (g(2))^2 = [5 \cdot 4 \cdot 2 - 3 \cdot 3] \div 4 = 31/4$.

(f) Let $Z(x) = g(x^2 + f(x))$. Compute $Z'(1)$.
Solution: Again by the chain rule and the product rule, $Z'(x) = g'(x^2 + f(x)) \cdot \frac{d}{dx}(x^2 + f(x)) = g'(x^2 + f(x)) \cdot (2x + f'(x))$, so $Z'(1) = g'(1 + f(1)) \cdot (2 + f'(1)) = g'(5) \cdot (2 + 6) = 1 \cdot 8 = 8$.

6. (10 points) Compute the following derivatives.

   (a) Let $f(x) = x + \sqrt{1+x^3}$. Find $\frac{d}{dx}f(x)$.
   Solution: Using the power rule and chain rule, $f'(x) = 1+\frac{1}{2}(1+x^3)^{-1/2}\cdot(3x^2)$.

   (b) Let $g(x) = \frac{x^3}{x^2+1}$. What is $g'(x)$?
   Solution: Use the quotient rule to get $g'(x) = 3x^2(1+x^2) - 2x(x^3) \div (1+x^2)^2 = \frac{x^4+3x^2}{(1+x^2)^2}$.

7. (10 points) Find two critical points of $h(x) = (x+2)\cdot(2x-1)^2$.

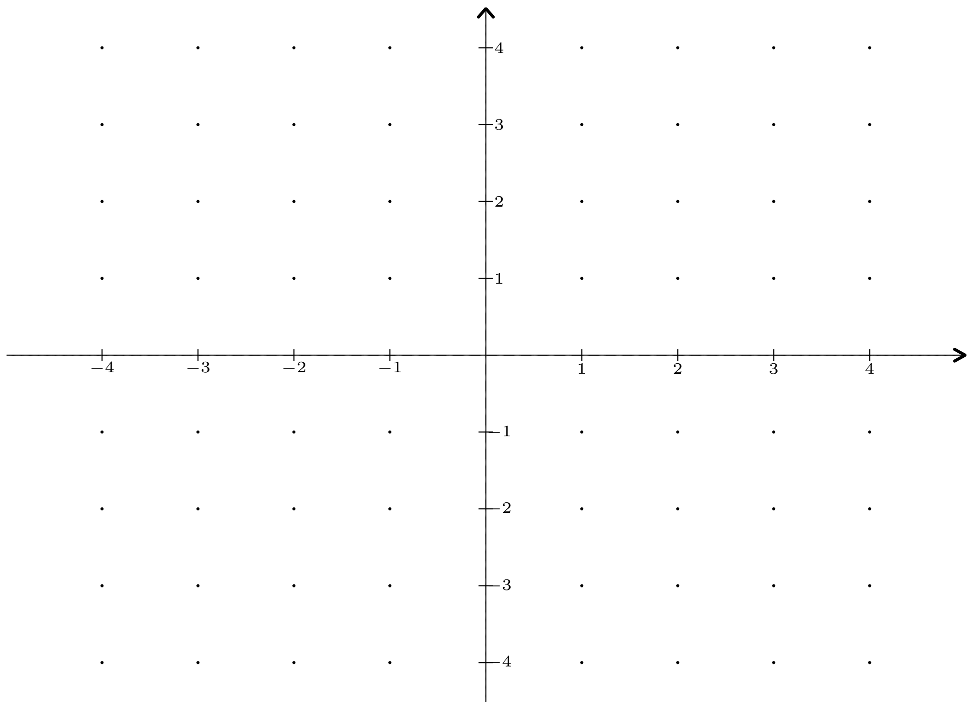
   Solution: By the product rule, $\frac{d}{dx}(x+2)\cdot(2x-1)^2 = 1(2x-1)^2 + 2(2x-1)\cdot 2\cdot(x+2)$. Simplifying, we have $h'(x) = (2x-1)[2x-1+4(x+2)] = (2x-1)(6x+7)$, which has two zeros, $x = \frac{1}{2}$ and $x = -\frac{7}{6}$.

8. (30 points) Consider the function

$$r(x) = \frac{(x^2-1)(3x+1)}{(2x^2-8)(x+1)}.$$

   Use the Test Interval Technique to find the sign chart of $r(x)$. Find the horizontal and vertical asymptotes, and sketch the graph of r. Your graph must be consistent with the information you find in the sign chart.

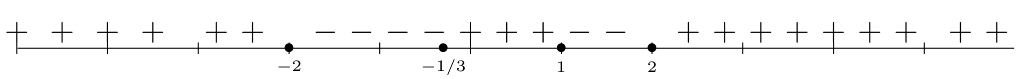
   Solution:



Solution: Notice first that r is not in factored form. Factoring reveals that the numerator and denominator have common factors. Thus

$$r(x) = \frac{(x-1)(x+1)(3x+1)}{2(x-2)(x+2)(x+1)}.$$

We can remove the common factor $x + 1$ with the understanding that we are (very slightly) enlarging the domain of r: $r(x) = \frac{(x-1)(3x+1)}{2(x-2)(x+2)}$. Next find the branch points. These are the points at which r can change signs. Precisely, they are the zeros of the numerator and of the denominator. They are $-1/3, 1, -2, 2$. The horizontal asymptote is $y = 3/2$, the vertical asymptotes are $x = 2$ and $x = -2$ and the zeros of r are $x = -1/3$ and $x = 1$. Again we select test points and find the sign of f at of these points to get the sign chart.



9. (7 points) Suppose $f(x)$ satisfies $f(3) = 2$ and the line tangent to the graph of f at the point $(3, 2)$ is $2y + 3x = 13$. What is $f'(3)$?

   Solution: The slope of the line is $-\frac{3}{2}$, so $f'(3) = -3/2$.

10. (15 points) Consider the function $h(x) = x^4 + 2x^3 - 12x^2 + 60x$. Find the

intervals over which h is concave upwards. Make clear which function you'‘re building the sign chart for and what the test points are.

Solution: Since $h'(x) = 4x^3 + 6x^2 - 24x + 60$ and $h'(x) = 12x^2 + 12x - 24 = 12(x + 2)(x - 1)$, we can see that $h''(x)$ is positive over $(-\infty, -2)$ and $(1, \infty)$, and negative on $(-2, 1)$, so h is concave upwards on $(-\infty, -2)$ and $(1, \infty)$.