**April 23, 2014** **Name** ______________________________

The problems count as marked. The total number of points available is 172. Throughout this test, **show your work.** Using a calculator to circumvent ideas discussed in class will generally result in no credit.

1. (10 points) Find an equation for the line tangent to the graph of $y = \ln(x^4+1)$ at the point $(1, \ln(2))$.

   **Solution:** The derivative of the function is $y' = 4x^3/(x^4+1)$ so the slope of the line at $x = 1$ is $\frac{4\cdot 1}{1^4+1} = 2$ and the line is $y - \ln(2) = 2(x-1)$. That is, $y = 2x - 2 + \ln(2)$.

2. (20 points) A function $g(x)$ has been differentiated to get

$$g'(x) = 2(x-3)^2 - 8.$$

   (a) Find the interval(s) over which $g'(x)$ is increasing.

   **Solution:** Since $g'(x)$ is a concave up quadratic polynomial with vertex $(3, -8)$, we conclude that $g'$ is increasing on $(3, \infty)$.

   (b) Find the interval(s) over which $g(x)$ is increasing.

   **Solution:** Solve $2(x-3)^3 - 8 = 0$ to find the two critical points of $g$, $x = 1$ and $x = 5$, and then build the sign chart for $g'$ to see that it's negative precisely on $(1, 5)$, so $g$ is increasing on $(-\infty, 1)$ and $(5, \infty)$.

   (c) Find the interval(s) over which $g(x)$ is concave upwards.

   **Solution:** Differentiate $g'$ to get $g''(x) = 2 \cdot 2(x-3)$ which is positive on $(3, \infty)$, so that is the interval where $g$ is concave upwards.

3. (15 points) Consider the function $f(x) = x^3 - 9x^2 + 24x$ on the interval $[0, 5]$.

   (a) What is the largest value of $f$ on its domain? In other words, find the absolute maximum of $f$ over $[0, 5]$.

   **Solution:** Find $f'$. $f'(x) = 3x^3 - 18x + 24 = 3(x^2 - 6x + 8) = 3(x-2)(x-4)$, so the critical points are $x = 2$ and $x = 4$. Checking the value of $f$ at the endpoints and the critical points, we have $f(0) = 0, f(2) = 20, f(4) = 16$ and $f(5) = 20$. So the maximum value of $f$ over the interval is 20.

   (b) What is the smallest value of $f$ on its domain? In other words, find the absolute minimum of $f$ over $[0, 5]$.

   **Solution:** The minimum value of $f$ is zero. See above.

4. (15 points) Consider the function $f(x) = \ln[(2x - 13)(3x - 4)^3\sqrt{x^2 + 3}]$.

   (a) Recall that $\ln(x)$ is defined precisely when $x > 0$. Find the domain of $f$.

   **Solution:** Build the sign chart for the function $g(x) = (2x - 13)(3x - 4)^3\sqrt{x^2 + 3}$ to see that $g$ is positive on $(-\infty, 4/3)$ and $(13/2, \infty)$. So the domain of the function $f$ is the union of these two sets.

   (b) Let $g(x) = (2x - 13)(3x - 4)^3\sqrt{x^2 + 3}$. Use logarithmic differentiation to find $g'$. Find a decimal representation of $g'(1)$.

   **Solution:** Take logs of both sides to get $\ln g(x) = \ln[(2x - 13)(3x - 4)^3\sqrt{x^2 + 3}]$. This simplifies to $\ln g(x) = \ln(2x - 13) + \ln(3x - 4)^3 + \frac{1}{2}\ln(x^2 + 3)$. Now taking the derivative of both sides yields

   $$\frac{g'(x)}{g(x)} = \frac{2}{2x - 13} + 3\frac{3}{3x - 4} + \frac{x}{x^2 + 3}.$$

   Finally, we can write $g'(x) = g(x)[\frac{2}{2x-13} + 3\frac{3}{3x-4} + \frac{x}{x^2+3}]$. Thus, $g'(1) = g(1)\left(\frac{2}{2-13} + \frac{9}{3-4} + \frac{1}{1+3}\right) = 22[-\frac{2}{11} - 9 + \frac{1}{4}] = -196.5$.

5. (20 points) Find all the critical points for each of the functions listed below.

   (a) $T(x) = 4(x^2 + 9)^{1/2} + 22 - 2x$.
   **Solution:** $T'(x) = 2(x^2 + 9)^{-1/2} \cdot 2x - 2$. Setting this equal to zero, we get $\frac{4x}{\sqrt{x^2+9}} = 2$, which yields two critical points $x = \pm\sqrt{3}$.

   (b) $f(x) = \ln(2x + 17) - 2x$.
   **Solution:** Note that $f'(x) = \frac{2}{2x+17} - 2$, so the critical is the solution to $4x + 34 = 2$, which is $x = -8$.

   (c) $g(x) = e^{x^2-4x}$.
   **Solution:** $g'(x) = e^{x^2-4x} \cdot (2x - 4)$, so $x = 2$ is the only critical point.

   (d) $h(x) = (x^2 - 4)^{2/3}$.
   **Solution:** $h'(x) = \frac{2}{3}(x^2 - 4)^{-1/3} \cdot 2x$, so the critical points are $x = 0$ (stationary) and $x = \pm 2$ (singular).

6. (20 points) A botanist conjectures that the height of a certain type of pine tree can be modeled by a learning curve. To test his conjecture, he plants a 2 foot tall tree. He knows that eventually the tree will grow to 40 feet tall, its maximum height. Suppose that after one year, the tree is 4 feet tall.

   (a) What does the model predict for the height of the tree after two years.

   **Solution:** We use the model $Q(t) = A - Be^{-kt}$ with the information that $Q(0) = 2$ and $\lim_{t\to\infty} Q(t) = 40$. So $A - B = 2$ and $A = 40$. Conclude that $B = 38$.

   (b) How many inches does the tree grow during the fourth year?

   **Solution:** Since the tree grows to 4 feet after one year, we have $Q(1) = 4 = 40 - 38e^{-k}$ which we solve for $k$ to get $k = \ln(19) - \ln(18) \approx 0.05406$. So the number of inches grown during the fourth year is $Q(4) - Q(3) = 38(e^{-3k} - e^{-4k}) \approx 1.7$ feet, or 20.4 inches.

   (c) What is the instantaneous rate of growth at $t = 3.5$ years.

   **Solution:** To find the instantaneous rate of growth at $t = 3.5$ years, differentiate $Q$. $Q'(t) = 38ke^{-kt}$ and at $t = 3.5$ is 1.7003 feet or 20.40 inches.

   (d) Describe the connection between the two answers (b) and (c).

   **Solution:** These two quantities are quite close because $(Q(4)-Q(3)/(4-3)$ is a good estimate of $Q'(3.5)$. Look at the graph to see just how close these two are.

7. (30 points) Suppose we know that the function $f$ has been differentiated and that $f'(x) = 2x(x^2 - 3)^4$. Also, the point $(2, 1/5)$ belongs to the graph of $f$.

   (a) Find an equation for the line tangent to the graph of $f$ at the point $(2, 1/5)$.

   **Solution:** Since $m = f'(2) = 2 \cdot 2 \cdot (x^2 - 3)^4 = 4$, the line is given by $y - 1/5 = 4(x - 2)$.

   (b) Find $f(1)$. Hint: $f$ is an antiderivative of $f'$.

   **Solution:** Anti-differentiating $f'(x)$ gives $f(x) = (x^2 - 3)^5/5 + C$. Evaluating $f(2) = 0.2 = 1^5/5 + C$ which happens only when $C = 0$. Thus $f(1) = (-2)^5/5 = -32/5$.

   (c) Find the area of the region $R$ bounded above by the graph of $f'(x)$, below, by the $x$-axis and on the sides by the lines $x = 0$ and $x = 1$.

   **Solution:** Just measure the growth of the antiderivative $f(x)$ of $f'(x)$ over the interval $(x^2 - 3)^5/5 \mid_0^1 = (-2)^5/5 - (-3)^5/5 = 211/5 = 42.2$.

8. (42 points) Compute each of the following integrals

   (a) $\displaystyle\int_1^2 \frac{(4x-5)^2}{x}\ dx$

   **Solution:** Rewrite the integrand to get $\displaystyle\int_1^2 \frac{(4x-5)^2}{x}\,dx = \int_1^2 \frac{16x^2-40x+25}{x}\,dx = \int_1^2 16x - 40 + \frac{25}{x}\ dx$. Thus, we have $8x^2 - 40x + 25\ln(x)|_1^2 \approx 1.328$.

   (b) $\displaystyle\int_0^1 \frac{d}{dx}(x^3 - 2x^2 + 7)\ dx$

   **Solution:** This is just $x^3 - 2x^2 + 7|_0^1 = -1$.

   (c) $\displaystyle\int_1^4 3x^2 e^{x^3}\ dx$

   **Solution:** $e^{x^3}|_1^4 = e^{64} - e^1$.

   (d) $\displaystyle\int_2^3 \frac{x^3 + 2x^2 - x}{x}\ dx$

   **Solution:** $\int (x^3 + 2x^2 - x)/x\ dx = \int x^2 + 2x - 1 dx = x^3/3 + x^2 - x|_2^3 = 34/3$.

   (e) $\displaystyle\int_1^3 \frac{2x+3}{x^2+3x-3}\ dx$

   **Solution:** By substitution, $(u = x^2 + 3x - 3)$, $\int \frac{2x+3}{x^2+3x-3}\ dx = \ln|x^2 + 3x - 3||_1^3 = \ln(15) \approx 2.71$.

   (f) $\displaystyle\int_{-1}^1 6x^5(x^6+3)^7\ dx$

   **Solution:** By substitution with $u = x^6 + 3$, $\int 6x^5(x^6+3)^7\ dx = \frac{(x^6+3)^8}{8}|_{-1}^1 = 0$.

   (g) $\displaystyle\int_1^2 (x-1)^5 x\ dx$

   **Solution:** By substitution with $u = x - 1, du = dx$, $\int (x-1)^5 x\ dx = \int u^5(u+1)\ du = \int u^6 + u^5\ du = \frac{1}{7}u^7 + \frac{1}{6}u^6 = \frac{1}{7}(x-1)^7 + \frac{1}{6}(x-1)^6|_1^2 = \frac{1}{7} + \frac{1}{6} = \frac{13}{42}$.