November 25, 2014 **Name** ______________________________

The total number of points available is 145. Throughout this test, **show your work.**

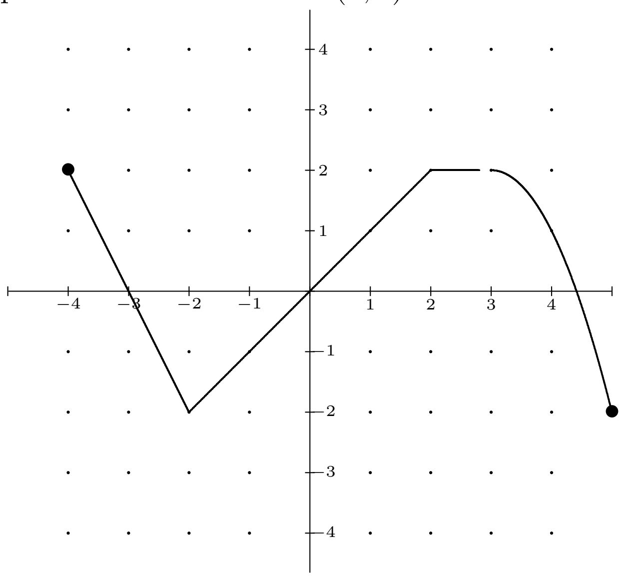
1. (10 points) Find an equation for the line tangent to the graph of $y = \ln(2x+1)$ at the point $(1, \ln(3))$.

   Solution: The derivative of the function is $y' = 2/(2x+1)$ so the slope of the line at $x = 1$ is $2/3$ and the line is $y - \ln(3) = 2(x-1)/3$. That is $y = \ln(3) + (2x-2)/3$.

2. (10 points) Find an equation for the line tangent to the graph of $y = e^{6x-2}$ at the point $(1/3, 1)$.

   Solution: $y' = 6(e^{6x-2})$ so $m = 6e^{6\cdot\frac{1}{3}-2} = 6$. Thus the line is $y - 1 = 6(x - 1/3)$. and in slope-intercept form $y = 6x - 1$.

3. (55 points) Consider the function f given below. The domain of f is $[-4, 5]$. The function is linear on each of the three intervals $[-4, -2]$, $[-2, 2]$ and $[2, 3]$. Over $[3, 5]$, it is a quadratic with vertex at $(3, 2)$.



(a) (20 points) Find a symbolic representation for f by filling in each clause in the function definition below:

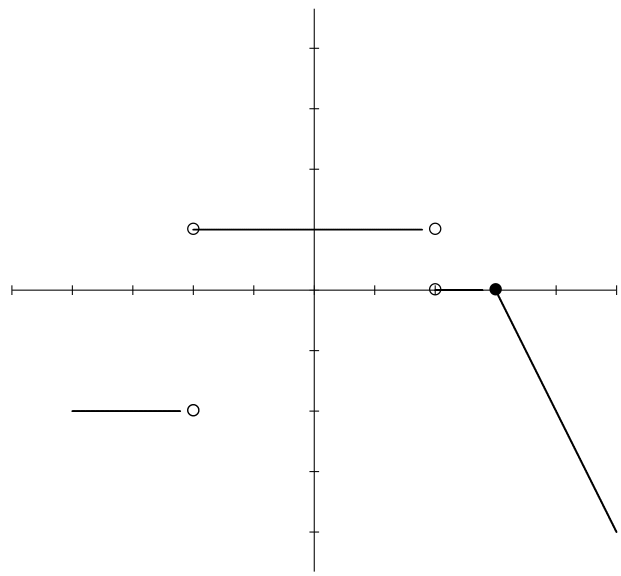
$$f(x) = \begin{cases} & \text{if } -4 \le x < -2 \\ & \text{if } -2 \le x < 2 \\ & \text{if } \quad 2 \le x < 3 \\ & \text{if } \quad 3 \le x \le 5 \end{cases}$$

Solution:

$$f(x) = \begin{cases} -2x - 6 & \text{if } -4 \le x < -2 \\ x & \text{if } -2 \le x < 2 \\ 2 & \text{if } \quad 2 \le x < 3 \\ -(x-3)^2 + 2 & \text{if } \quad 3 \le x \le 5 \end{cases}$$

(b) (15 points) On the axes provided, sketch the graph of f'.

Solution:



(c) Find three whole number values of x where the function f is increasing.

Solution: $-1, 0, 1$

(d) Find two whole number values of x where f' does not exist.

Solution: $-2, 2$. Note that $f'(3) = 0$.

(e) Find three whole number values of x where f' is positive.

Solution: $-1, 0, 1$

(f) Find three whole number values of x where f'' has the value 0.

Solution: $-3, -1, 0, 1$

4. (10 points) A skull from an archeological dig has one-twelfth the amount of Carbon-14 it had when the specimen was alive.

   (a) Recall that the half-life of Carbon-14 is 5770 years. Find the decay constant k.

   Solution: We must solve the equation $Q(5770) = Q_0/2$ for k, where $Q(t) = Q_0 e^{-kt}$. The equation leads to $e^{-5770k} = 1/2$ which means that $k \approx 1.201 \times 10^{-4} = 0.00012$.

   (b) What is the age of the specimen? Round off your answer to the nearest multiple of one hundred years.

   Solution: To solve the equation $Q_0 e^{-kt} = Q_0/12$, divide both sides by Q_0 to get $e^{-kt} = 1/12$. This happens for $t = 20685$ years, which rounds to 20700.

5. (10 points) How long does it take an investment of \$$P$ at an annual rate of 8% to triple in value if compounding

   (a) takes place quarterly? Round your answer to the nearest tenth of a year.

   Solution: We need to solve the equation $3P = P\left(1 + \frac{0.08}{4}\right)^{4t} = P(1.02)^{4t}$ for t. Using logs, we get $t = \frac{\ln 3}{4 \ln 1.02} \approx 13.87 \approx 13.9$ years.

   (b) takes place continuously? Round your answer to the nearest tenth of a year. As usual, no work shown, no credit!

   Solution: We need to solve the equation $3000 = 1000e^{0.08t}$ for t. Using logs, we get $t = \frac{\ln 3}{0.08} \approx 13.73 \approx 13.7$ years.

6. (20 points) A botanist conjectures that the height of a certain type of pine tree can be modeled by a learning curve. To test his conjecture, he plants a 2 foot tall tree. He knows that eventually the tree will grow to 40 feet tall, its maximum height. Suppose that after one year, the tree is 4 feet tall.

   (a) What does the model predict for the height of the tree after two years.

   Solution: We use the model $Q(t) = A - Be^{-kt}$ with the information that $Q(0) = 2$ and $\lim_{t\to\infty} Q(t) = 40$. So $A - B = 2$ and $A = 40$. Conclude that $B = 38$.

   (b) How many inches does the tree grow during the fourth year?

   Solution: Since the tree grows to 4 feet after one year, we have $Q(1) = 4 = 40 - 38e^{-k}$ which we solve for k to get $k = \ln(19) - \ln(18) \approx 0.05406$. So the number of inches grown during the fourth year is $Q(4) - Q(3) = 38(e^{-3k} - e^{-4k}) \approx 1.7$ feet, or 20.4 inches.

   (c) What is the instantaneous rate of growth at $t = 3.5$ years.

   Solution: To find the instantaneous rate of growth at $t = 3.5$ years, differentiate Q. $Q'(t) = 38ke^{-kt}$ and at $t = 3.5$ is 1.7003 feet or 20.40 inches.

   (d) Describe the connection between the two answers (b) and (c).

   Solution: These two quantities are quite close because $(Q(4) - Q(3)/(4 - 3)$ is a good estimate of $Q'(3.5)$. Look at the graph to see just how close these two are.

A. (15 points) Consider the function $f(x) = (x^2 - 4x + 4)e^{2x}$.

(a) Use the product rule to find $f'(x)$.
Solution: $f'(x) = (2x - 4)e^{2x} + 2(x^2 - 4x + 4)e^{2x}$

(b) List the critical points of f.
Solution: Factor the expression above to get $(2x^2 - 8x + 8 + 2x - 4)e^{2x} = 2e^{2x}(x - 2)(x - 1)$, which has value 0 when $x = 2, x = 1$.

(c) Construct the sign chart for $f'(x)$.
Solution: f' is positive on $(-\infty, 1)$ and on $(2, \infty)$.

(d) Write in interval notation the interval(s) over which f is increasing.
Solution: f is increasing on $(-\infty, 1)$ and on $(2, \infty)$.

B. (15 points) Consider the function $f(x) = \ln[(2x^2+3)(7x-2)(x^2-4)]$.

(a) Recalling that $\ln(x)$ is defined precisely when $x > 0$, find the domain of f.

Solution: Build the sign chart for the function $g(x) = (2x^2+3)(7x-2)(x^2-4) = (2x^2+3)(7x-2)(x-2)(x+2)$ to see that g is positive on $(-2, 2/7)$ and $(2, \infty)$. So the domain of the function f is the union of these two sets.

(b) Let $g(x) = (2x^2+3)(7x-2)^2(x^2-4)^3$. Use logarithmic differentiation to find g'. You need not simplify your answer.

Solution: Take logs of both sides to get $\ln g(x) = \ln(2x^2+3)(7x-2)^2(x^2-4)^3$. This simplifies to $\ln g(x) = \ln(2x^2+3) + 2\ln(7x-2) + 3\ln(x^2-4)$. Now taking the derivative of both sides yields

$$\frac{g'(x)}{g(x)} = \frac{4x}{2x^2+3} + \frac{2 \cdot 7}{7x-2} + \frac{3 \cdot 2x}{x^2-4}$$

Finally, we can write

$$g'(x) = (2x^2+3)(7x-2)^2(x^2-4)^3\left(\frac{4x}{2x^2+3} + \frac{14}{7x-2} + \frac{6x}{x^2-4}\right).$$

(c) Find an equation for the line tangent to g at the point $(-1, g(-1))$.

Solution: First, $g(-1) = 5 \cdot 81 \cdot (-27) = -10935$. The $g'(-1) = -10935(-\frac{16}{45}) = 3888$, so the line is $y + 10935 = 3888(x+1)$.