April 8, 2015 **Name** ______________________________

The total number of points available is 168. Throughout this test, **show your work.** **Throughout this test, you are expected to use calculus to solve problems. Graphing calculator solutions will generally be worth substantially less credit.**

1. (12 points) Find an equation for the line tangent to the graph of $f(x) = xe^{-2x+4}$ at the point $(2, f(2))$.

   Solution: Find f' first. Then note that $f'(2) = 1 + 2(-2) \cdot 1 = -3$ and $f(2) = 2$, so the line is $y = -3x + 8$.

2. (12 points) In 1985, the tuition at Yale University was \$10000 per year. In 2015 it was about \$44000 per year. Estimate the annual percent growth. Write a sentence to justify your answer.

   Solution: We can write $44000 = 10000e^{30r}$, assuming continuous compounding, and a 30 year time frame. Solve this by taking the log of both sides to get $30r = \ln(4.4)$. This yields $r \approx 4.938$ percent growth.

3. (12 points) Find an equation for the line tangent to the graph of $f(x) = x^2 \ln(x)$ at the point $(1, f(1))$.

   Solution: First $f'(x) = 2x \ln(x) + x^2/x$. Then note that $f'(1) = 0 + 1 = 1$, so the line is $y - 0 = 1(x - 1)$.

4. (30 points) There is a function g whose derivative is given below:

$$g'(x) = \begin{cases} x^2 - 2x - 3 & \text{if } -4 \leq x \leq 4 \\ 9 - x & \text{if } 4 < x \leq 12 \end{cases}$$

(a) What is the domain of g'. Use interval notation.

Solution: The domain of g' is $[-4, 12]$.

(b) Find the critical points of g'.

Solution: To find the critical points of g' we need to examine g''.

$$g''(x) = \begin{cases} 2x - 2 & \text{if } -4 < x < 4 \\ -1 & \text{if } 4 < x < 12 \end{cases}$$

So we can see that $x = 1$ is a stationary point and $x = 4$ is a singular point.

(c) Find the intervals over which g' is decreasing.

Solution: By inspection, we see that g' is decreasing on both $[-4, 1]$ and $[4, 12]$. Alternatively, we could note that g'' is negative on these two intervals.

(d) Find the intervals over which the function g is decreasing.

Solution: The function g is decreasing precisely when its derivative g' is negative. That happens for the two intervals $(-1, 3)$ and $(9, 12)$.

(e) Find the critical points of g.

Solution: $g'(x) = 0$ for $x = -1, x = 3$ and $x = 9$.

(f) Find the absolute maximum and absolute minimum of g'. You must show all your work.

Solution: We must compare the values of g' at the endpoints -4 and 12 with its value at the critical points 1 and 4. Note that $g'(-4) = 16 + 8 - 3 = 21$, $g'(12) = -3$, $g'(1) = -4$ and $g'(4) = 16 - 8 - 3 = 5$, so we can see that g' has its absolute maximum at $x = -4$ and absolute minimum at $x = 1$.

5. (20 points) Consider the function $f(x) = x^4$. In this problem we are looking for the point on the graph of f that is closest to the point $(0, 16)$. We'll prove that the point exists as follows. Note that the points belonging to the graph of f are of the form $(x, y) = (x, x^4)$. Build the distance function $d(x)$ that measures the distance from $(0, 16)$ to (x, x^4). For example $d(2)$ is the distance between $(0, 16)$ and $(2, 2^4)$, which is just 2.

   (a) Let $D(x)$ be the square of $d(x)$. In other words, D is the inside part of d, but without the radical. Compute $D'(x)$.

   Solution: First note that $d(x) = \sqrt{x^2 + (x^4 - 16)^2}$. Hence $D(x) = x^2 + (x^4 - 16)^2$ and $D'(x) = 2x + 2(x^4 - 16) \cdot 4x^3$.

   (b) Notice that $x = 0$ is a critical point. Is it the location of a relative maximum, a relative minimum, or an imposter? Write a sentence supporting your answer.

   Solution: $D'(x) = 2x + 2(x^4 - 16) \cdot 4x^3 = 2x[1 + 2(x^4 - 16) \cdot 2x^2]$, so $x = 0$ is a critical point. Since $D'(x) > 0$ to the immediate left of 0 and negative just the right of 0, it follows that $x = 0$ is a local maximum.

   (c) What is $D'(1)$? What is $D'(2)$? Since D' is continuous, you can apply the Intermediate Value Theorem. Is this critical point a relative max or min, or neither. Is $D(x)$ increasing or decreasing at $x = 2$?

   Solution: Note that $D'(1) < 0$ and $D'(2) > 0$. Since D is a polynomial, it follows that D' is continuous, and therefore, by IVT it must have a zero in the interval $(1, 2)$. Since the sign of D' changes from negative to positive, the critical point must be a local minimum.

6. (15 points) For each function f listed below, find the slope of the line tangent to its graph at the point $(0, f(0))$.

   (a) $f(x) = e^{e^x}$.
   Solution: $f'(x) = e^{e^x} \cdot e^x$, so $f'(0) = e^{e^0} \cdot e^0 = e$.

   (b) $f(x) = (x-1)^2 \cdot \ln(2x+1)$.
   Solution: $f'(x) = 2(x-1) \cdot \ln(2x+1) + \frac{2}{2x+1}(x-1)^2$, so $f'(0) = 2(0-1) \cdot \ln(1) + \frac{2}{1}(0-1)^2 = 2$.

   (c) $f(x) = (1 + \ln(2x+1))^3$.
   Solution: $f'(x) = 3(1+\ln(2x+1))^2 \cdot (0+\frac{2}{2x+1})$, so $f'(0) = 3(1+0)^2(2) = 6$.

7. (10 points) For each function listed below, find a critical point.

   (a) $g(x) = 4\sqrt{x^2+1} - 2x + 20$.
   Solution: $g'(x) = 4\frac{2x}{2\sqrt{x^2+1}} - 2$, so $g'(x) = 0$ if $4x = 2(\sqrt{x^2+1}$. Squaring both sides we have $16x^2 = 4x^2 + 4$ and then we get $x = \pm\sqrt{1/3}$.

   (b) $h(x) = (2x-3)e^{4x}$.
   Solution: $h'(x) = 2e^{4x} + 4e^{4x}(2x-3) = e^{4x}(2 + 8x - 12)$. Therefore, we have $8x - 10 = 0$ and $x = 1.25$.

8. (20 points) Consider the function $f(x) = \frac{(2x+3)(x-3)}{x(x-1)}$.

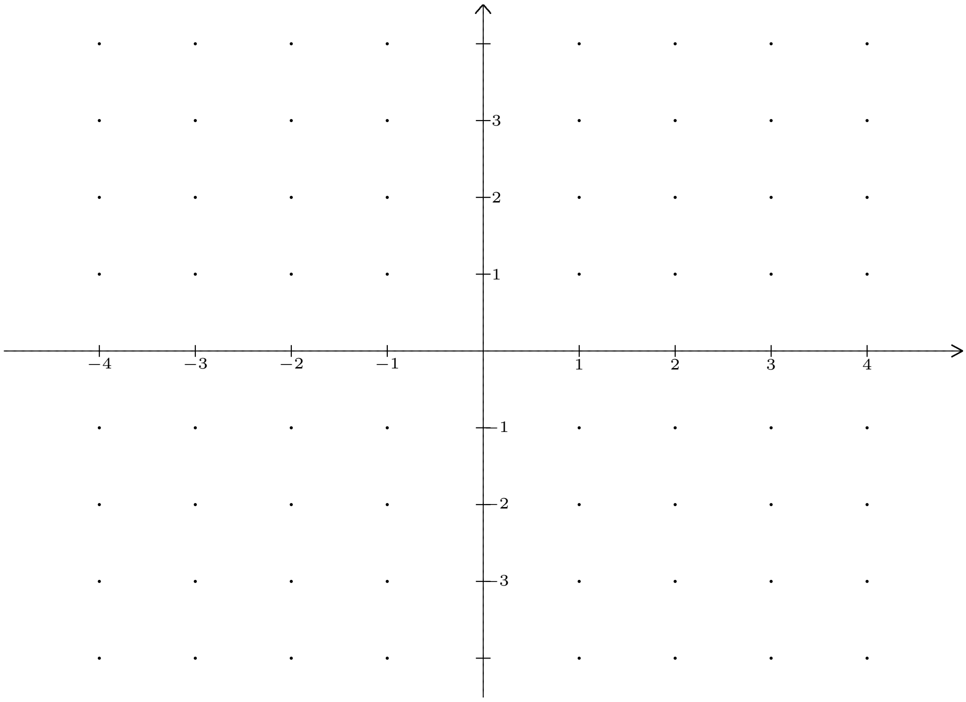
   (a) Build the sign chart for f

   Solution: We have to use all the points where f could change signs, $x = -3/2, 3, 0$, and 1. As expected the signs alternate starting with + at the far left: $+ - + - +$.

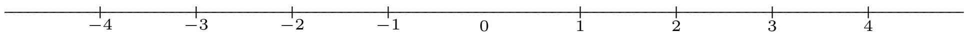
   (b) Find the vertical and horizontal asymptotes and the zeros, being careful not to mix them up.

   Solution: The zeros are $x = -3/2$ and $x = 3$ and the vertical asymptotes are $x = 0$ and $x = 1$. The horizontal asymptote is $y = 2$.

   (c) Use the information from the first two parts to sketch the graph of f.



   (d) From the graph, you can speculate on the existence of critical points if there are any. Write a sentence about where you expect to find these critical points or why you think there are none. Estimate the sign chart for $r'(x)$



Solution: Based on the graph, f has one critical point and it is in the interval $(0, 1)$. Suppose it is α. The f' is negative on $(-\infty, 0)$, negative on $(0, \alpha)$, positive on $(\alpha, 1)$, and positive on $(1, \infty)$.

9. (12 points) Compound Interest.

   (a) Consider the equation $2000(1 + 0.03)^{4t} = 6000$. Find the value of t and interpret your answer in the language of compound interest.

   Solution: t is the time required for an investment at rate $r = 12\%$ compounded quarterly to triple. Use logs to get $t = 9.29$ years.

   (b) Consider the equation $P(1 + 0.04)^{4\cdot 10} = 5000$. Solve for P and interpret your answer in the language of compound interest.

   Solution: P is the principle in dollars required to grow a 16% investment compounded quarterly over 10 years to grow to \$5000. Another way to say this is that P is the present value of \$5000 compounded quarterly over 10 years. Solve the equation to get $P = \$1041.45$

   (c) Consider the equation $Pe^{10r} = 2P$. Solve for r and interpret your answer in the language of compound interest.

   Solution: We're compounding continuously, and getting twice the original investment. If we interpret the r as rate, we're asking what rate of interest will cause a continuously compounded 10-year investment to double. Solve $10r = \ln 2$ to get $r = 0.069$ or 6.9%.

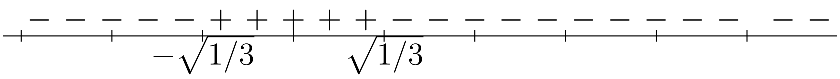
10. (25 points) Consider the function $f(x) = \ln(3x^2 + 1)$.

    (a) Find $f'(x)$.
    Solution: $f'(x) = \frac{6x}{3x^2+1}$.

    (b) Find an equation for the line tangent to the graph of f at the point $(3, f(3))$.
    Solution: Since $f'(3) = 18/28 = 9/14$ and $f(3) = \ln 28$, we have $y - \ln 28 = 9(x - 3)/14$.

    (c) Find $f''(x)$.
    Solution: $f''(x) = \frac{6(3x^2+1)-6x(6x)}{(3x^2+1)^2}$.

    (d) Find the sign chart for $f''(x)$.
    Solution: $f''(x) < 0$ on $(-\infty, -\sqrt{1/3})$ and on $(\sqrt{1/3}, \infty)$ and positive on $(-\sqrt{1/3}, \sqrt{1/3})$, as shown on the sign chart for f'':



    (e) Find the intervals over which f is concave upwards.
    Solution: From (c) it follows that f is concave upwards on $(-\sqrt{1/3}, \sqrt{1/3})$.