June 21, 2001 **Name** ______________________________

The total number of points possible is 118. **SHOW YOUR WORK**

1. (10 points) . Find the relative maxima and relative minima, if any, of $g(x) = x^2+16/x^2+4$. Demonstrate that you understand either the first derivative test or the second derivative test that distinguishes relative maxima from relative minima.

   Solution: Note that $g'(x) = 2x-32x^{-3}$ and $g''(x) = 2+96x^{-4}$. The stationary points are ± 2. There are no singular points, but I will not take away points if you claim that 0 is a singular point (g is not defined at 0 so it does not qualify to be a singular point. Apply either the first derivative test or the second to find that g has minimums at both 2 and -2. Note that $g''(2) = g''(-2) > 0$.

2. (20 points) Suppose you have differentiated a function $f(x)$ and found that

$$f'(x) = \frac{(x-4)(x+3)^2}{(x-2)(3x)(x+5)}.$$

   (a) Find the intervals over which f is increasing.

   Solution: The critical numbers are the zeros of the numerator and the denominator, namely, $x = 4, -3, 2, 0$, and 5. Use the test interval technique to find that f is increasing over the intervals $(-\infty, -5), (0, 2), (4, \infty)$.

   (b) Find an equation for the horizontal asymptote of the function f', if there is one.

   Solution: There is one horizontal asymptote, which is the ration of the coefficients of the cubic terms, $y = 1/3$.

   (c) Find equations for all vertical asymptotes of the function f'.

   Solution: The vertical asymptotes occur, according to the asymptote theorem, at the zeros of the denominator of the reduced rational function. Thus, $x = 2, x = 0$, and $x = -5$.

3. (10 points) Let $f(x) = \frac{1}{2}x^4 + x^3 - 6x^2 + 3x - 2$.

   (a) Find the interval(s) where f is concave upward.

   Solution: First note that $f'(x) = 2x^3 + 3x^2 - 12x + 3$ and $f''(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x-1)(x+2)$, so $f''(x) = 0$ for the two values $x = 1$ and $x = -2$. Use the test interval technique on to find the intervals where $f'' > 0$. You find that f is concave upward on the two intervals $(-\infty, -2)$ and $(1, \infty)$.

   (b) Find the inflection points of f, if there are any.

   Solution: There are two inflection points, $(-2, f(-2) = (-2, -32)$ and $(1, f(1)) = (1, -7/2)$.

4. (10 points) Solve each of the equations below for x in terms of the other letters.

   (a) $4a \cdot b^{2x} = \sqrt{a}$

   Solution: Use the laws of logarithms to find that

$$x = \frac{-\log 4 + (1/2)\log a}{2\log b}.$$

   (b) $\frac{a}{1+b^x} = b^4$

   Solution: Again, use the laws of logarithms to find that

$$x = \frac{\log(ab^{-4} - 1)}{\log b}.$$

   (c) $4e^{2x-3} = 28$.

   Solution: Again, use the laws of logarithms to find that

$$x = \frac{\ln 7 + 3}{2} \approx 2.4729.$$

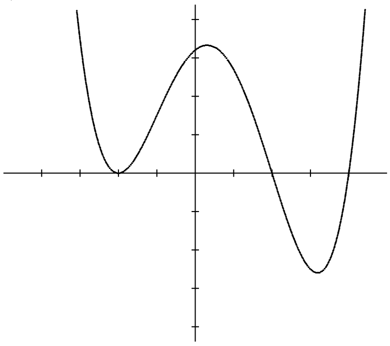
5. (8 points) Find the rate of change of $s(t) = e^{3t} \cdot \ln t$ when $t = 1$.

   Solution: Differentiate to get $s'(t) = 3e^{3t} \cdot \ln t + 1/t \cdot e^{3t}$ and evaluate this at $t = 1$ to get $s'(1) = 3e^3 \cdot \ln 1 + 1 \cdot e^3 = e^3 \approx 20.08$.

6. (12 points) A radioactive substance has a half-life of 28 years. Find an expression for the amount of the substance at time t if 30 grams were present initially.

   Solution: Recall that the model is given by $Q(t) = Ae^{-kt}$. We use the given data to solve for A and k as follows: $Q(0) = Ae^{-k\cdot 0} = A = 30$, and $Q(28) = .5 \cdot 30 = 15 = 30 \cdot e^{-28t}$ which translates into $e^{-28k} = 1/2$. Taking the natural logs of both sides gives $k = \frac{-\ln 2}{-28} \approx 2.475 \cdot 10^{-2} = 0.02475$. Thus, $Q(t) = 30e^{0.02475t}$.

7. (16 points) Questions (a) through (d) refer to the graph of the fourth degree polynomial function f given below.



   (a) The number of roots of $f''(x) = 0$ is

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

   Solution: Since the graph is concave down from about -1 to about 2 and concave up on both sides of this interval, there must be two points of inflection, ie. two roots of $f''(x) = 0$.

   (b) The number of roots of $f'(x) = 1$ is

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

   Solution: Stare at the graph until you see that there are 3 places where the tangent line has a slope of 1.

   (c) The number of roots of $f(x) = 1$ is

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

   Solution: Draw the horizontal line $y = 1$ and notice that the line hits the graph at four places.

(d) A good estimate of $f'(2)$ is

(A) -2 **(B)** 0 **(C)** 1 **(D)** 1.8 **(E)** 3.2

Solution: Notice that the line tangent to f at $x = 2$ has negative slope, so the answer can only be -2.

8. (12 points) An amount of \$1000 is invested at an interest rate of 9 percent per year with interest compounded a. monthly and b. continuously?

   (a) How long does it take the monthly compounded account to double in value?

   Solution: We're solving the equation $2 = 1 \cdot (1 + \frac{0.09}{12})^{12t}$ for t. Take logs of both sides to find $t = \frac{\log 2}{12 \log 1.0075} \approx 7.7304$ years. This rounds off to $t = 7.7$ years.

   (b) How long does it take the continuously compounded account to triple in value? Express your answer to the nearest tenth of a year.

   Solution: Here we're solving $3 = 1 \cdot e^{0.09t}$. Taking the natural log of both sides get $t = \frac{\ln 3}{0.09} \approx 12.2$ years.

9. (20 points) Compute the following derivatives.

   (a) Find f' when $f(x) = x^3 \cdot e^{2x}$.
   Solution: By the product rule, $f'(x) = 3x^2 e^{2x} + 2e^{2x} \cdot x^3$.

   (b) Find g' when $g(x) = \ln(2x^3)$.
   Solution: By the chain rule, $g'(x) = \frac{6x^2}{2x^3} = \frac{3}{x}$.

   (c) Find f' when $f(x) = x \ln x - x$.
   Solution: By the product rule, $f'(x) = 1 \ln x + x \cdot \frac{1}{x} - 1 = \ln x$.

   (d) Find f' when $f(x) = e^{x^3}$.
   Solution: By the chain rule, $f'(x) = 3x^2 \cdot e^{x^3}$.

   (e) Find f' when $f(x) = x^3/e^{2x}$.
   Solution: By the quotient rule, $f'(x) = \frac{3x^2 e^{2x} - 2x^3 e^{2x}}{e^{4x}}$.