November 15, 2001 Your name ____________________________

The multiple choice count 5 points each and the true-false problems count 3 points each. In the multiple choice section, circle the correct choice (or choices). You must show your work on the other problems. The total number of points available is 122.

1. True-false questions. Circle the appropriate word, true or false.

   (a) True or false. Every continuous function f on an interval $[a, b]$ must have an absolute maximum somewhere in the interval. Ie, there exists a point $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$.

   Solution: True. This is just what the BIG theorem says.

   (b) True or false. The graph of a function cannot touch or intersect a horizontal asymptote to the graph of f.

   Solution: False. There is nothing in the definition of horizontal asymptote that implies this.

   (c) True or false. If $f'(c) = 0$, then f has a relative maximum or a relative minimum at $x = c$.

   Solution: False. The function can have neither a max nor a min at a stationary point. Look at $f(x) = x^3$ and 0.

   (d) True or false. If f has a relative maximum or a relative minimum at $x = c$, then $f'(c) = 0$.

   Solution: False. All we can tell is that c is a critical point. It might be a singular point.

   (e) True or false. If $f'(c) = 0$ and $f''(c) < 0$, then f has a relative maximum at $x = c$.

   Solution: True. This is just the second derivative test.

2. A stone is thrown straight upward from the roof of an 80-foot high building. The height in feet of the stone at any time t (in seconds) is given by $h(t) = -16t^2 + 64t + 80$.

   (a) How many seconds elapse before the stone hits the ground?

   (A) 2 seconds **(B)** 3 seconds **(C)** 4 seconds
   (D) 5 seconds **(E)** 6 seconds

   Solution: D. Solve the equation $h(t) = 0$ to get $-16(t^2 - 4t - 5) = 0$ and $t = -1, t = 5$. Only $t = 5$ makes any sense.

   (b) At what time does the stone reach its maximum height?

   (A) 1 second **(B)** 3/2 seconds **(C)** 2 seconds
   (D) 5/2 seconds **(E)** 3 seconds

   Solution: C. Differentiate h and set h' equal to zero to get $-32t+64 = 0$, so $t = 2$ seconds.

3. Which of the following is a horizontal asymptote of $r(x)$? Circle all those that apply.

$$r(x) = \frac{(x+4)(x^2-1)(3x^2-4)}{(x^2+x-12)(x-1)^4}$$

   (A) $x = 1$ **(B)** $x = 3$ **(C)** $y = 0$ **(D)** $y = 1$ **(E)** $y = 3$

   Solution: C. Note that the degree of the numerator is 5 while the degree of the denominator is 6. Thus $y = 0$ is the horizontal asymptote.

4. Referring again to the function $r(x)$ in the previous problem, which of the following is a vertical asymptote. Again circle all that apply.

   (A) $y = 0$ **(B)** $x = -4$ **(C)** $x = -1$ **(D)** $x = 1$ **(E)** $x = 3$

   Solution: D and E. First, factor both numerator and denominator and cancel the common factors. When this is done, you see that the factor $x+4$ disappears from the denominator, leaving just $x - 1$ and $x - 3$.

On all the following questions, **show your work.**

5. (10 points) Sketch the graph of a function f with domain $[-4, -2) \cup (-2, 4]$ on the coordinate axes provided that has all the following properties:

   (a) $f(-4) = f(-1) = 0$ and $f(0) = -1$.

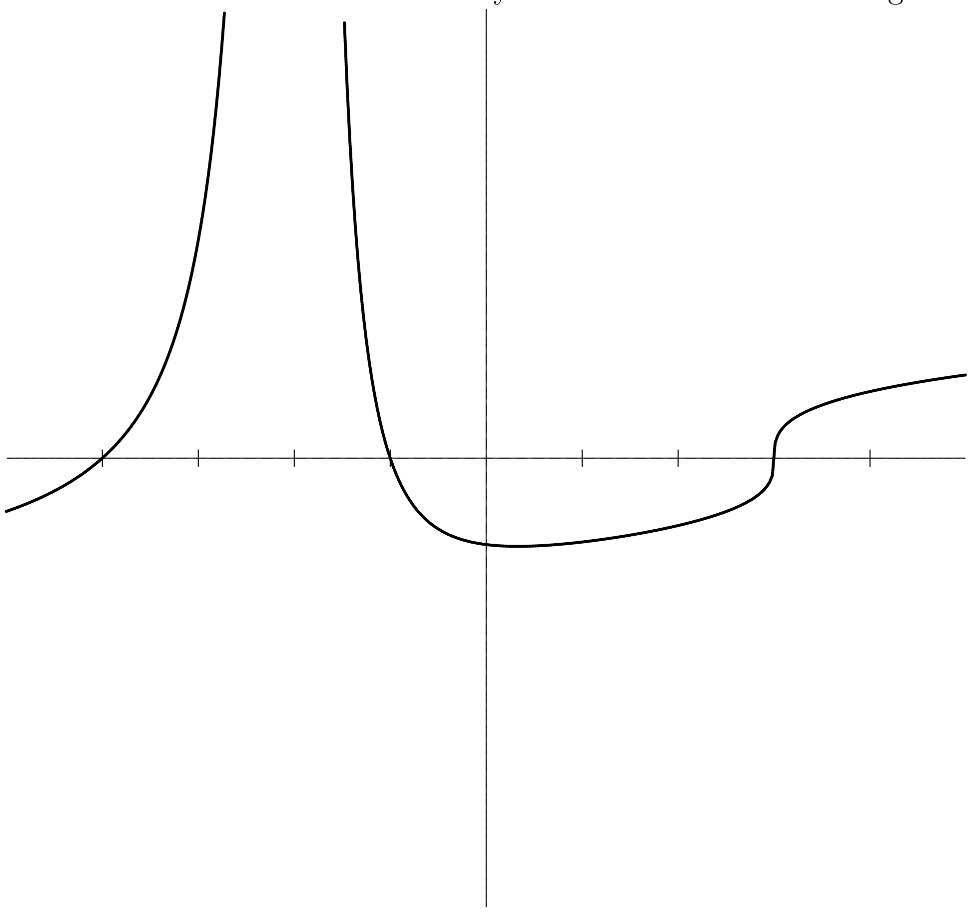
   (b) f has a stationary point (ie, $f' = 0$) at $x = 1/2$.

   (c) f has a singular point (ie, f' is undefined) at $x = 3$.

   (d) f has a local minimum at $1/2$.

   (e) f has a vertical asymptote at $x = -2$.

   Solution: Of course there are many solutions. One solution is given below.



6. (40 points) Let $g(x) = (2x - 3)^3(x + 1)^2$.

   (a) Find $g'(x)$ and the critical points of g. Express g' in factored form.

Solution: Use the product theorem to get $g'(x) =$

$$\begin{aligned} 3(2x-3)^2 \cdot 2(x+1)^2 + 2(x+1)(2x-3)^3 &= \\ (x+1)(2x-3)^2[6(x+1)+2(2x-2)] &= \\ (x+1)(2x-3)(6x+6+4x-6) &= \\ (x+1)(2x-3)^2(10x) \end{aligned}$$

which has three zeros, $x = -1, x = 3/2$, and $x = 0$.

(b) Find $g''(x)$, and express it in factored form. Use the second derivative test and other methods to classify the points found in part a as relative minimums, relative maximums, or neither.

Solution: There are several ways to find $g''(x)$, one of which is to rewrite $g'(x)$ as $g'(x) = 10(x^2+x)(2x-3)^2$ and to use the product rule. This gives $g''(x) =$

$$\begin{aligned} 10[(2x+1)(2x-3)^2 + (x^2+x)\cdot 2(2x-3)\cdot 2] &= \\ 10(2x-3)[(2x+1)(2x-3)+4x^2+4x] &= \\ 10(2x-3)[4x^2-4x-3+4x^2+4x] &= \\ 10(2x-3)(8x^2-3) &= \end{aligned}$$

which has three zeros, $x = 3/2$ and $x = \pm\sqrt{\frac{3}{8}}$.

(c) Use the test interval technique to determine the intervals over which g is increasing.

Solution: We are solving the inequality $g'(x) > 0$. So our line is divided into four intervals by the critical points of g. The intervals are $(-\infty, -1)$, $(-1, 0)$, $(0, 3/2)$, and $(3/2, \infty)$. Checking the value of g' at test points in these four intervals, we find that $g'(x)$ is positive on the first, third and fourth intervals. Thus g is increasing on $(-\infty, -1)$ and $(0, \infty)$ since the third and fourth intervals can be merged.

(d) Use the test interval technique to determine the intervals over which g is concave upwards

Solution: We are solving the inequality $g''(x) > 0$. So our line is divided into four intervals by the critical points of g', ie by the zeros of g''. The intervals are $(-\infty, \sqrt{\frac{3}{8}})$, $(-\sqrt{\frac{3}{8}}, \sqrt{\frac{3}{8}})$, $(\sqrt{\frac{3}{8}}, 3/2)$ and $(3/2, \infty)$. Checking the value of g'' at test points in these four intervals, we find that $g''(x)$ is positive on the second and fourth intervals. Thus g is concave upwards on $(-\sqrt{\frac{3}{8}}\sqrt{\frac{3}{8}})$ and $(3/2, \infty)$.

7. (20 points) A topless box is constructed from a rectangular piece of cardboard that measures 16 inches by 12 inches. An x by x square is cut from each of the four corners, and the sides are then folded upwards to build the box.

   (a) Express the volume V as a function of x.
   Solution: $V(x) = x(16 - 2x)(12 - 2x)$

   (b) Use the physical constraints to find the domain of V.
   Solution: $0 \leq x \leq 6$.

   (c) Find the derivative of V and use it to find the critical points of V.
   Solution: Rewrite $V(x)$ as $V(x) = (16x - 2x^2)(12 - 2x)$ and use the product rule to get $V'(x) =$

$$\begin{aligned} (16-4x)(12-2x) + 16x - 2x^2)(-2) &= \\ 8x^2 - 80x + 12 \cdot 16 + 4x^2 - 32x &= \\ 12x^2 - 112x + 12 \cdot 16 &= \\ 4(3x^2 - 28x + 48). \end{aligned}$$

   (d) Compute the value of V at all the points in the domain, including endpoints, where V could have an absolute extrema.
   Solution: Use the quadratic formula to get

$$x = \frac{28 \pm \sqrt{28^2 - 4 \cdot 3 \cdot 48}}{6} = \frac{28 \pm \sqrt{208}}{6} \approx 7.07, \text{ and } 2.2629,$$

   only the later of which is in the domain of V.

   (e) What is the volume of the largest box that can be so constructed?
   Solution: $V(2.2629) \approx 194.067$ is the absolute maximum. The absolute minimum ($= 0$) occurs at the endpoints, 0 and 6.

8. (16 points) Optimal Charter Flight Fare. If exactly 160 people sign up for a charter flight, the agency charges \$300. However, if more than 160 sign up, the agency reduces the fare by \$0.80 for each additional person.

   (a) Let x denote the number of passengers beyond 160. Construct the revenue function $R(x)$.

   Solution: $R(x) = (160 + x)(300 - 0.8x)$.

   (b) Find all the critical points of your revenue function.

   Solution: $R'(x) = (300 - 0.8x) - 0.8(160 + x) = -1.6x + 172$, which is zero when $x = 107.5$, which is the location of a relative maximum of R.

   (c) What number of passengers results in the maximum revenue?

   Solution: The optimal number of passengers to take is $160 + 107.5 = 267.5$ (maybe 267 adults and a baby).

   (d) What is the maximum revenue?

   Solution: The optimal revenue is $R(107.5) = (160{+}107.5)(300{-}0.8(107.5)) = 57,245$.