November 23, 2015 **Name** ____________________

The problems count as marked. The total number of points available is 130. Throughout this test, **show your work.** Use of calculator to circumvent ideas discussed in class will generally result in no credit.

1. (50 points) You must show your work.

   (a) (10 points) Let $F(x) = (x^2 - 3x + 1)e^{-x}$. Find the intervals where F is increasing.

   Solution: Note that $F'(x) = (2x - 3)e^{x^2-3x+1} + (x^2 - 3x + 1)(-1)e^{-x}$, which can be factored to give $F'(x) = -e^{-x}(x^2 - 5x + 4)$, so the function is increasing on the interval $[1, 4]$ where its derivative is nonnegative.

   (b) (10 points) In the year 2000 the population of the earth was estimated to be 6.5 billion people. Today the estimate is 7.3 billion. What is the annual population growth? Round your answer to the nearest hundredth of a percent.

   Solution: Thinking of the year 2000 as the initial value $Q(0)$, we have $Q(t) = 6.5e^{kt}$ and $Q(15) = 7.3 = 6.5e^{15k}$, which we can solve to get $k = \ln(7.3/6.5) \approx 0.0077 = .77\%$, a little less than one percent.

   (c) (10 points) Let $G(x) = x\ln(x^2 - 3)$. Find an equation for the line tangent to G at the point $(2, G(2))$.

   Solution: First, $G'(x) = \ln(x^2 - 4) + \frac{x \cdot 2x}{x^2-3}$ by the product rule. Thus $G'(2) = 0 + 8 = 8$ and $G(2) = \ln(1) = 0$. So the line we want is $y = 8x - 16$.

   (d) (10 points) Let $K(x) = x^2e^{-2x}$. Discuss the concavity of K.

   Solution: The second derivative of K is given by $K''(x) = (4x^2 - 8x + 2)e^{-2x}$, which has two zeros. $1 \pm \frac{\sqrt{2}}{2}$, and we can see that G is concave upwards on the interval these two points determine.

   (e) (10 points) Let $L(x) = (e^{-2x} + 1)^3$. Find an equation for the line tangent to L at the point $(0, 8)$.

   Solution: Note that $L'(x) = 3(e^{-2x}+1)^2(-2e^{-2x})$, so $L'(0) = 3 \cdot 2^2(-2) = -24$, and the line we want is $y = -24x + 8$.

2. (20 points) Let's assume that American male height can be modeled using a learning curve. Such a person is about 1 foot long at birth and about 6 feet tall at age 60, and they do not get any taller.

   (a) Solve for two of the three parameters.

   Solution: Note that $Q(t) = A - Be^{-kt}$, $Q(0) = 1$, and

   (b) $lim_{t\to\infty} Q(t) = 6$. From this it follows that $A = 6$ and $B = 5$.

   (c) If a child is 18 inches long at age 1, how many years will it take until he is 5 feet tall?

   Solution: We have $Q(1) = 6 - 5e^{-k} = 3/2$ from which it follows that $k = -\ln(0.9) \approx 0.105$. Then $Q(15.27) \approx 5$.

   (d) How much growth does the person exhibit during the two year period from age 9 to age 11?

   Solution: $Q11 - Q(9) \approx 4.43 - 4.06 \approx 0.37$ ft, which is about 4.4 inches.

   (e) Find $Q'(10)$ and compare this with your result in part (c). Explain in English what you think?

   Solution: $Q'(t) = -5(-k)e^{-kt} = 5ke^{-kt}$, so $Q'(10) \approx 5(0.105)(9/10)^1 0 \approx$ 0.184ft which is roughly 2.2 inches, nearly exactly half the answer above. The reason is that $Q'(10) \approx \frac{Q(11)-Q(9)}{11-9}$.

3. (30 points) Let

$$f'(x) = \begin{cases} -x-5 & \text{if } x < 0 \\ -(x-3)(x-5) & \text{if } 1 \le x \end{cases},$$

(a) What is the domain of f'? Express your answer in interval notation.

Solution: The domain of f' is $(-\infty, 0) \cup [1, \infty)$.

(b) Find the intervals over which f' is increasing.

Solution: f' is increasing on $[0, 4]$.

(c) Find the intervals over which the function f is increasing.

Solution: f is increasing precisely where f' is positive, $(-\infty, -5)$ and $(3, 5)$.

(d) Suppose $f(2) = 5$. Find an equation for the line tangent to the graph of f at the point $(2, 5)$.

Solution: Since $f'(2) = -(2-3)(2-5) = -3$, we have $y - 5 = -3(x-2)$.

4. (15 points) Let $H(x) = \ln((x^2 - 4)(x^2 - 16))$.

   (a) Recall that $\ln(x)$ is defined precisely when $x > 0$. What is the domain of H (in interval notation)?
   Solution: H is defined precisely when $(x^2 - 4)(x^2 - 16) > 0$, which is $(-\infty, -4) \cup (-2, 2) \cup (4, \infty)$.

   (b) Find the slope of the line tangent to H at the point $x = 1$.
   Solution: Note that $\frac{H'(x)}{H(x)} = \frac{2x}{x^2-4} + \frac{2x}{x^2-16}$, so $H'(1) = \ln(45)\left(\frac{2}{-3} + \frac{2}{-15}\right) = -\frac{4}{5}\ln(45)$.

5. (15 points) Build a function that satisfies the logistic curve model

$$Q(t) = \frac{A}{1 + Be^{-kt}}$$

   that satisfies

   - $Q(0) = 100$
   - $lim_{t\to\infty} Q(t) = 150$
   - $Q'(0) = 20$

   Then find a value of t for which $Q(t) = 125$.

   Solution: Note that $\frac{A}{1+B} = 100$ and $A = 150$. So $B = 1/2$. Solve $Q'(0) = 20$ to get $k = 0.4$. Then $\frac{150}{1+.5e^{\cdot 4t}} = 125$ results in $e^{-0.4t} = 0.4$ and $t = 2.29$.

6. (10 points) Find a cubic polynomial with a max at $x = -3$, a min at $x = 2$ and has value 2 at $x = 0$.

   Solution: Let $p'(x) = (x+3)(x-2) = x^2 + x - 6$, so $p(x)$ could be $\frac{x^3}{3} + \frac{x^2}{2} - 6x + 2$.

7. (10 points) Find an equation for the line tangent to the graph of $g(x) = e^{e^x}$ at the point $x = \ln(2)$.

   Solution: $g'(x) = e^{e^x} \cdot e^x$ and $g'(\ln(2)) = 2e^2$, so we have $y - e^2 = 2e^2(x - \ln(2))$

8. (10 points) The number a satisfies $2^a = 7$. Estimate 9^a correct to three significant places.

   Solution: Since $a = \frac{log7}{log2}$, $9^a = 9^{log7/log2} \approx 477$.

9. (10 points) What is the y-intercept of the line tangent to the graph of $f(x) = (1 + e^{-2x+4})^2$ at the point $(2, f(2))$.

   Solution: Find f' first: $f'(x) = 2(1 + e^{-2x+4})(e^{-2x+4})(-2)$. Then note that $f'(2) = 2(1 + e^0)(e^0)(-2) = -8$ and $f(2) = 4$, so the line is $y = -8x + 20$.