March 27, 2002 Your name ________________

The first 7 problems count 5 points each. Problems 8 through 11 are multiple choice and count 7 points each and the final ones counts as marked. In the multiple choice section, circle the correct choice (or choices). You do not need to show your work on problems 8 through 11, but you must show your work on the other problems. The total number of points available is 158.

Each of the next few items are true-false. To get full credit you must give a valid reason for your answer. Circle either True or False, and give your reason in the space provided. Generally, 2 points for the right t/f value and 3 points for the right reason.

1. True or false. If $f'(x) > 0$ for each x in the interval $(-1, 1)$, then f is increasing on $(-1, 1)$.

   Solution: True, by theorem A.

2. True or false. If $f''(x) < 0$ on the interval (a, c) and $f''(x) > 0$ on the interval (c, b), then the point $(c, f(c))$ is a point of inflection of f.

   Solution: True. This is basically the definition of inflection point.

3. True or false. If $f'(c) = 0$, then f has a relative maximum or a relative minimum at $x = c$.

   Solution: False. The function can have neither a max nor a min at a stationary point. Look at $f(x) = x^3$ and $x = 0$.

4. True or false. If f has a relative maximum at $x = c$, then $f'(c) = 0$.

   Solution: False. All we can tell is that c is a critical point. It might be a singular point.

5. True or false. If $f'(c) = 0$ and $f''(c) < 0$, then f has a relative maximum at $x = c$.

   Solution: True. This is just the second derivative test.

6. True or false. If f and g are differentiable, then $\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$.

   Solution: False. Look up the product rule.

7. True or false. If $h(x) = \sqrt{x^2 - 4}$, then $h'(x) = \frac{1}{2}(x^2 - 4)^{-1/2}$.

   Solution: False. Look up the chain rule. In fact, $h'(x) = \frac{1}{2}(x^2 - 4)^{-1/2} \cdot 2x$.

The next few problems are multiple choice. There is no need to justify your answers.

8. Over which of the intervals is the function $f(x) = x^4 - 4x^3 - 18x^2 + 2x + 1$ concave down?

   (A) $[-2, 0]$ **(B)** $[-1, 2]$ **(C)** $[1, 4]$ **(D)** $[2, 6]$ **(E)** $[3, 8]$

   Solution: B. Note that $f'(x) = 4x^3 - 12x^2 - 36x + 2$ and that $f''(x) = 12x^2 - 24x - 36 = 12(x^2 - 2x - 3) = 12(x + 1)(x - 3)$, so there are two points of inflection. By the test interval technique, $f''(x) > 0$ throughout the interval $[-1, 3]$. The only option that is inside this interval is B. $[-1, 2]$.

9. The line tangent to the graph of $g(x) = 3x^2 - 5x$ at the point $(1, -2)$ has a y-intercept of

   (A) -3 **(B)** -2 **(C)** -1 **(D)** 1 **(E)** 2

   Solution: A. Note that $g'(x) = 6x - 5$, so $g'(1) = 6 - 5 = 1$. The tangent line has slope 1 and passes through the point $(1, -2)$. Thus $y + 2 = 1(x - 1)$ which is the same as $y = x - 3$.

10. The absolute maximum value of the function $f(x) = 2x^3 - 9x^2 + 12x + 4$ on the interval $-2 \leq x \leq 2$ is

    (A) -10 **(B)** 0 **(C)** 9 **(D)** 10 **(E)** 12

    Solution: C. Find $f'(x)$ first and then the critical points that are between -2 and 2. $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$, so there one critical point and two endpoints to check. $f(-2) = -72; f(1) = 9; f(2) = 8$, so the absolute maximum is $f(1) = 9$, which of course occurs at $x = 1$.

11. The absolute maximum value value of the function $f(x) = 2x^3 - 9x^2 + 12x + 4$ on the interval $-2 \leq x \leq 3$ is

    (A) -10 **(B)** 0 **(C)** 9 **(D)** 12 **(E)** 13

    Solution: E. Find $f'(x)$ first and then the critical points that are between -2 and 3. $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$, so there are two critical points and two endpoints to check. $f(-2) = -72; f(1) = 9; f(2) = 8$; and $f(3) = 13$, so the absolute maximum is $f(3) = 13$, which of course occurs at $x = 3$.

On all the following questions, **show your work.**

12. (12 points) Let $g(x) = (2x-6)^2(x+3)^2$.

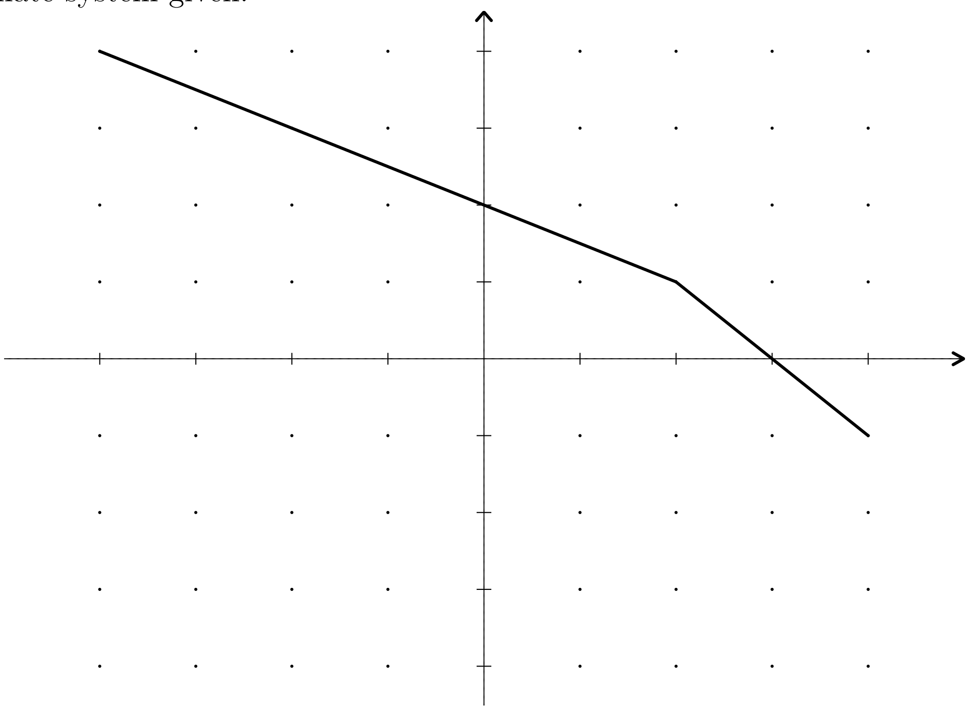
    (a) Use the test interval technique (not a graphing calculator) to find the intervals over which g is increasing.

    Solution: First differentiate g by the product rule: $g'(x) = 2(2x-6)2(x+3)^2 + 2(x+3)^2(2x-6) = 2(2x-6)(x+3)(2(x+3)+(2x-6)) = 2(2x-6)(x+3)4x$. So there are three critical points, $x = 3; x = -3$; and $x = 0$. Evaluating g' at test points, $-4, -2, 2$ and 4 shows that $g'(x) \geq 0$ on the intervals $[-3, 0]$ and $[3, \infty)$, so g is increasing precisely on those two intervals.

    (b) Find and classify each critical point as a location of a. a relative maximum, b. a relative minimum, or c. neither a relative max nor a relative min.

    Solution: Based on what we did above, we can apply the first derivative test to learn that g has relative mins at -3 and 3 and a relative max at 0.

13. (8 points) Sketch and example of a continuous function $f(x)$ that has a domain of $[-4, 4]$, and has singular point at $x = 2$ and a value of 1 at $x = 2$, but does not have a relative maximum or a relative minimum at $x = 2$. Use the coordinate system given.

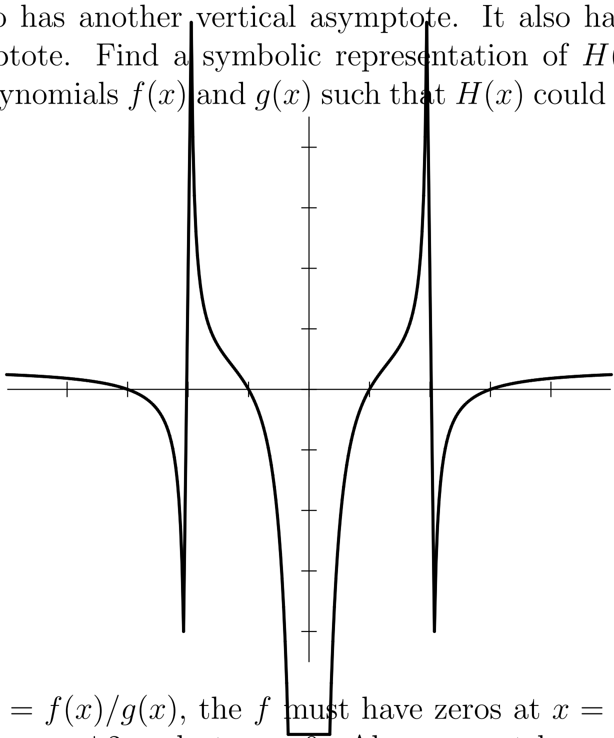


Solution: A pair of linear functions pieced together works well here. Let

$$f(x) = \begin{cases} -x/2 + 2 & \text{if } x \leq 2 \\ x + 3 & \text{if } x > 2 \end{cases}$$

Notice that f does not have a tangent line at $x = 2$ and it does not have a relative extremum there either.

14. (15 points) The function $H(x)$ shown below is a rational function with four zeros, $x = \pm 1$ and $x = \pm 3$, and vertical asymptotes at $x = -2$ and $x = 2$. Notice that H also has another vertical asymptote. It also has $y = 1/2$ as a horizontal asymptote. Find a symbolic representation of $H(x)$. In other words, find two polynomials $f(x)$ and $g(x)$ such that $H(x)$ could be $f(x)/g(x)$.



Solution: If $H(x) = f(x)/g(x)$, the f must have zeros at $x = \pm 1, \pm 3$ and g must have zeros at $x = \pm 2$ and at $x = 0$. Also, g must be a polynomial of degree 4 since f is such a polynomial. Thus we can try $f(x) = (x-1)(x+1)(x-3)(x+3)$ and $g(x) = x^2(x-2)(x+2)$. This doesn't give H a horizontal asymptote of $y = 1/2$ however. Inserting a 2 in front of g does the trick. Thus

$$H(x) = \frac{(x^2-1)(x^2-9)}{2(x^2)(x^2-4)}.$$

15. (15 points) Consider the rational function

$$f(x) = \frac{(x^2-4)(x+1)}{(2x^2-3)(x-2)}.$$

(a) Find the horizontal asymptote(s).

Solution: The coefficient of x^3 in the numerator is 1 while that in the denominator is 2, so $y = 1/2$ is the horizontal asymptote.

(b) Find the vertical asymptotes.

Solution: To find the vertical asymptotes, you must first reduce the fraction to lowest terms, which mean cancelling out the common factors, in this case, just the $x-2$'s. This results in a denominator that has value 0 only at $x = \pm\sqrt{3/2}$, so these are the two vertical asymptotes.

(c) Compute $\lim_{x\to\infty} f(x)$.

Solution: The limit in question is the same as the horizontal asymptote, 1/2.

16. (15 points) The quantity demanded per month, x of a certain brand of electric shavers is related to the price, p, per shaver by the equation $p = -0.2x + 1000 \quad (0 < x < 20,000)$, where p is measured in dollars. The total monthly cost for manufacturing the shavers is given by $C(x) = 0.0001x^3 - 0.3x^2 + 1000x + 2000$. Recall that the revenue is the product of the demand and the price per unit. Construct the revenue function, $R(x)$. How is the profit related to revenue and cost? Find $P'(x)$, where $P(x)$ denotes the profit function. How many shavers should be produced per month in order to maximize the company's profit? What is the maximum profit?

Solution: First, the revenue function is $R(x) = x \cdot p(x) = x(-0.2x+1000)$ and the profit function is given by $P(x) = R(x) - C(x)$. Thus $P(x) = x(-0.2x + 1000) - (0.0001x^3 - 0.3x^2 + 1000x + 2000)$, and $P'(x) = -0.4x + 1000 - 0.0003x^2 + 0.6x - 1000$. Combining terms and simplifying yields $P'(x) = 0.2x - 0.0003x^2$, which leads to the critical point $x = 667$, and a maximum profit of $12,814.80$. You could also use the number 666.667.

This part of the test is for you to take home. Its due next class meeting. You're on your honor not to discuss it with anyone. You must sigh a pledge that you have not done so. Sign below: I have neither given nor received help on either of these problems.

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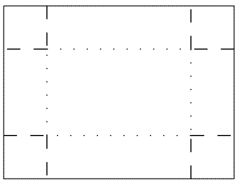
For both problems below, let F be the number of letters in your first (= given) name, and let L be the number of letters in your last (= family) name. The point is to customize the problems for you.

17. (15 points) Four congruent $x \times x$ squares from the corners of a cardboard rectangle that measures $2F \times 2L$. The sides are then folded upward to form a topless box. Find the volume V as a function of x. What is the logical domain? Compute $V(0), V(1), V(2)$, and $V(3)$. Find $V'(x)$ and use this to determine the critical points of V. Find the absolute maximum value of V and the value of x where it occurs.



Solution: The following solution is meant to satisfy everyone. Your solution will not look like this because your F and L are determined by your name. However, in order to make the solution relevant to everyone, I'm using F and L throughout. First $V(0) = 0, V(1) = 1(2F-2)(2L-2), V(2) = 2(2F-4)(2L-4)$, and $V(3)+3(2F-6)(2L-6)$. More generally, $V(x) = x(2F-2x)(2L-2x) = 4x(F-x)(L-x)$. Therefore

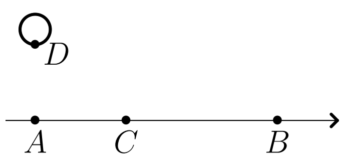
$$\begin{aligned}
V'(x) &= 4(F-x)(L-x) + 4x(F-x)(-1) + 4x(L-x)(-1) \\
&= 4(F-x)(L-x) - 4x(F-x) - 4x(L-x) \\
&= 4[(F-x)(L-x) - x(F-x) - x(L-x)] \\
&= FL - Fx - Lx + x^2 - Fx + x^2 - Lx + x^2 = FL - 2Fx - 2Lx + 3x^2.
\end{aligned}$$

Thus we need to solve $3x^2 - (2F+2L)x + FL = 0$. By the quadratic formula,

$$x = \frac{2F + 2L \pm \sqrt{4F^2 + 8FL + 4L^2 - 4 \cdot 3FL}}{6}$$

$$
\begin{aligned}
&= \frac{2F + 2L \pm 2\sqrt{F^2 - FL + L^2}}{6} \\
&= \frac{2F + 2L \pm 2\sqrt{F^2 - FL + L^2}}{6} \\
&= \frac{F + L \pm \sqrt{F^2 - FL + L^2}}{3}
\end{aligned}
$$

18. (15 points) You're on an island at the point D that is F miles from the closest on shore point A. The point A is $2L$ miles from the point B where you need to get as soon as possible. You can swim at 2 miles per hour and you can run at 6 miles per hour. You decide to swim to a point C between A and B that is x miles from A, then run from C to B. What is the distance from D to C as a function of x? What is the distance from C to B as a function of x? Find the time it takes to make the swim-run journey if $x = 0$ miles, $x = 1$ mile, and $x = 2$ miles. Then find the time $T(x)$ required to make the trip when x is any reasonable value. What is the domain of T? What are the critical points of T? What is the absolute minimum value of $T(x)$?



Solution: Let $d_1 = \sqrt{F^2 + x^2}$ and $d_2 = 2L - x$. The distance from D to B going through C is the sum $d_1 + d_2 = \sqrt{F^2 + x^2} + 2L - x$ and the time required is $T(x) = \frac{\sqrt{F^2+x^2}}{2} + \frac{2L-x}{6}$. Then $T'(x) = \frac{1}{2}\frac{1}{2}(F^2 + x^2)^{-1/2} \cdot 2x - \frac{1}{6}$. We set $T'(x)$

equal to zero to find the critical points of T. To solve $\frac{x}{2\sqrt{F^2-x^2}} = \frac{1}{6}$, cross multiply, square both sides, then collect like terms to get $9x^2 = F^2 + x^2$ from which it follows that $x = \pm\frac{F}{\sqrt{8}}$, only the positive value of which is relevant. Thus, $x = \frac{F\sqrt{2}}{4}$ and the time required to make the trip is $T(\frac{F\sqrt{2}}{4}) = \frac{F\sqrt{2}+L}{3}$.