November 29, 2017 **Name** ______________________________

The total number of points available is 163. Throughout this test, **SHOW YOUR WORK**. Throughout this test, you are expected to use calculus to solve problems. Graphing calculator solutions will generally be worth substantially less credit.

1. (12 points) Find an equation for the line tangent to the graph of $f(x) = xe^{-2x+4}$ at the point $(2, f(2))$.

   Solution: Note that $f'(x) = e^{-2x+4} - 2xe^{-2x+4}$, so $f'(2) = 1 + 2(-2) \cdot 1 = -3$ and $f(2) = 2$, so the line is $y - 2 = -3(x - 2)$ or $y = -3x + 8$.

2. (12 points) In 1985, the tuition at Yale University was \$10000 per year. In 2015 it was about \$44000 per year. Estimate the annual percent growth. Write a sentence to justify your answer.

   Solution: We can write $44000 = 10000e^{30r}$, assuming continuous compounding, and a 30 year time frame. Solve this by taking the log of both sides to get $30r = \ln(4.4)$. This yields $r \approx 4.938$ percent growth.

3. (12 points) Find an equation for the line tangent to the graph of $f(x) = x^2 \ln(x)$ at the point $(1, f(1))$.

   Solution: First $f'(x) = 2x\ln(x) + x^2/x$. Then note that $f'(1) = 0 + 1 = 1$, so the line is $y - 0 = 1(x - 1)$.

4. (40 points) There is a function g whose derivative is given below:

$$g'(x) = \begin{cases} x+17 & \text{if } -10 \le x \le -3 \\ x^2 - 3x - 4 & \text{if } -3 < x \le 6 \\ 14 & \text{if } 6 < x \le 10 \end{cases}$$

(a) What is the domain of g'. Use interval notation.
Solution: The domain of g' is $[-10, 10]$.

(b) Find the critical points of g'.
Solution: To find the critical points of g' we need to examine g''.

$$g''(x) = \begin{cases} 1 & \text{if } -10 < x < -3 \\ 2x - 3 & \text{if } -3 < x < 6 \\ 0 & \text{if } 6 < x < 10 \end{cases}$$

So we can see that $x = 3/2$ is a stationary point and $x = -3, 6$ are singular points.

(c) Find the intervals over which g' is increasing.
Solution: By inspection, we see that g' is increasing on both $[-10, -3]$ and $[3/2, 6]$. Alternatively, we could note that g'' is positive on these two intervals.

(d) Find the intervals over which the function g is decreasing.
Solution: The function g is decreasing precisely when its derivative g' is negative. That happens for the interval $(-1, 4)$.

(e) Find the critical points of g.
Solution: $g'(x) = 0$ for $x = -1, x = 3$ and $x = 9$.

(f) Find the absolute maximum and absolute minimum of g'. You must show all your work.
Solution: We must compare the values of g' at the endpoints -4 and 12 with its value at the critical points 1 and 4. Note that $g'(-4) = 16 + 8 - 3 = 21$, $g'(12) = -3$, $g'(1) = -4$ and $g'(4) = 16 - 8 - 3 = 5$, so we can see that g' has its absolute maximum at $x = -4$ and absolute minimum at $x = 1$.

5. (42 points) Exponentials and Logarithms.

   (a) You invest \$1000 at 8% for three years compounded quarterly. How much more would the final amount be if the investment is compounded continuously.

   Solution: $1000e^{3(0.08)} - 1000\left(1 + \frac{0.08}{4}\right)^{4\cdot 3} = \$1271.25 - \$1268.24 = \$3.01.$

   (b) Suppose the half life of a radioactive substance is one year. Find the decay constant k.

   Solution: Solve $Q_0/2 = Q_0 e^{-1k}$ to get $-k = \ln(1/2) = -\ln(2)$, so $k \approx 0.693$.

   (c) Suppose the time required for a continuously compounded investment to triple is 12 years. What is the time required to double.

   Solution: Since $3P = Pe^{12r}$, it follows that $r = \ln(3)/12$ and the doubling time is $t = \ln(2)/r = \frac{\ln(2)}{\ln(3)} \cdot 12 \approx 7.57$ years.

   (d) Suppose $Q(t) = \frac{A}{1+Be^{-kt}}$ is used to model the growth of a rumor in a population of 1000 where t is measured in hours. The rumor starts at a party attended by 100 people. After one hour, 200 people have heard the rumor. What is k?

   Solution: Since the limiting value of $Q(t)$ is 1000, it follows that $A = 1000$. Since $Q(0) = 100 = 1000/(1 + Be^{-k}$, it follows that $B = 9$. Since $Q(1) = 200$, it follows that $9e^{-k} = 4$, and we have $k = \ln(9) - \ln(4) \approx 0.811$.

   (e) Solve the equation $\ln(x + 2) - \ln(4x + 3) = \ln(1/x)$.

   Solution: First note that $\ln(\frac{x+2}{4x+3} = \ln(1/x)$, from which it follows that $x(x + 2) = 4x + 3$. Solve the resulting quadratic for x to get $x = -1$ and $x = 3$. Of course, only $x = 3$ works.

   (f) Solve for x: $2e^{2x} - 11e^x + 15 = 0$.

   Solution: Factor to get $(2e^x - 5)(e^x - 3) = 0$, so $x = \ln(2.5)$ and $x = \ln(3)$.

6. (20 points) Consider the function $f(x) = \frac{(2x^2+4)(x-3)}{x(x-1)}$. Follow the steps below to build the line tangent to f at the point $(4, f(4))$. This procedure is called logarithmic differentiation.

   (a) Let $G(x) = \ln(f(x))$. Find $G'(x)$.
   Solution: Since $G(x) = \ln\left(\frac{(2x^2+4)(x-3)}{x(x-1)}\right) = \ln(2x^2+4) + \ln(x-3) - \ln(x) - \ln(x-1)$, it follows that $G'(x) = \frac{4x}{2x^2+4} + \frac{1}{x-3} - \frac{1}{x} - \frac{1}{x-1}$.

   (b) Note that $G'(x) = \frac{f'(x)}{f(x)}$. What is $f(4)$?
   Solution: $f(4) = 36/12 = 3$.

   (c) Note that $G'(4) = f'(4)/f(4)$. Use this fact to find $f'(4)$.
   Solution: Since $G'(4) = \frac{16}{36} + \frac{1}{1} - \frac{1}{4} - \frac{1}{3}$, it follows that $f'(4) = 3(\frac{31}{36}) = \frac{31}{12}$.

   (d) Build the line tangent to f at $(4, f(4))$
   Solution: $y - 3 = \frac{31}{12}(x - 4)$.

7. (25 points) Consider the function $f(x) = \ln(3x^2 + 1)$.

   (a) Find $f'(x)$.
   Solution: $f'(x) = \frac{6x}{3x^2+1}$.

   (b) Find an equation for the line tangent to the graph of f at the point $(3, f(3))$.
   Solution: Since $f'(3) = 18/28 = 9/14$ and $f(3) = \ln(28)$, we have $y - \ln(28) = 9(x - 3)/14$.

   (c) Find $f''(x)$.
   Solution: $f''(x) = \frac{6(3x^2+1)-6x(6x)}{(3x^2+1)^2}$.

   (d) Find the sign chart for $f''(x)$.
   Solution: $f''(x) < 0$ on $(-\infty, -\sqrt{1/3})$ and on $(\sqrt{1/3}, \infty)$ and positive on $(-\sqrt{1/3}, \sqrt{1/3})$, as shown on the sign chart for f'':

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$-0.577 \qquad 0.577$

   (e) Find the intervals over which f is concave upwards.
   Solution: From (c) it follows that f is concave upwards on $(-\sqrt{1/3}, \sqrt{1/3})$.