Math 1120 Section 4 **Calculus** **Test 3**

October 29, 2004 **Name** ______________________________

The total number of points available is 135. Throughout this test, **show your work.**

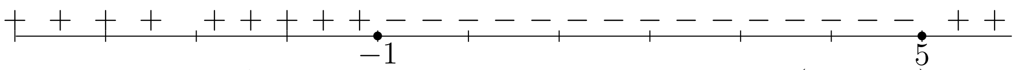
1. (25 points) Let $g(x) = x^3 - 6x^2 - 15x + 32$.

   (a) Find the critical points of g.

   Solution: Compute $g'(x)$ and find its zeros. $g'(x) = 3x^2 - 12x - 15 = 3(x^2 - 4x - 5) = 3(x+1)(x-5)$, so the stationary points of g are $x = -1$ and $x = 5$. There are no singular points.

   (b) Find the intervals over which g is increasing.

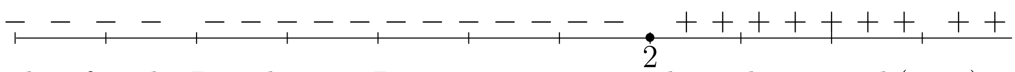
   Solution: The sign chart for g' is given below.



   By Big Theorem A, it follows that g is increasing on both $(-\infty, -1)$ and $(5, \infty)$.

   (c) Find the intervals over which g is concave upward.

   Solution: The sign chart for $g''(x) = 6x - 12$ is given below.



   Therefore, by Big Theorem B, g is concave upwards on the interval $(2, \infty)$.

   (d) Find the locations of local maxima and minima for g.

   Solution: There is one local maximum, and it occurs at $x = -1$ (note that $g''(-1) < 0$). There is one local minimum at $x = 5$ (note that $g''(5) > 0$). Finally, $g(-1) = 40$ and $g(5) = -68$.

   (e) What is the maximum value of g over the interval $[0, 10]$?

   Solution: Compare the values of g at the critical points and the end-points. $g(0) = 32, g(5) = -68, g(10) = 282$ so the maximum value of g in the interval is $g(10) = 282$.

2. (20 points) Find the critical points of each function.

   (a) $f(x) = (x-4)^2(2x-3)^3$

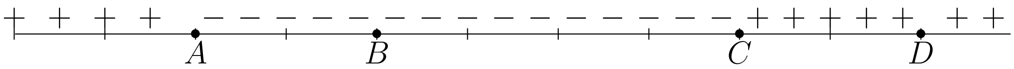
   Solution: Use the product rule to get $f'(x) = 2(x-4)(2x-3)^3 + 3(2x-3)^2) \cdot 2(x-4)^2 = (x-4)(2x-3)^2[2(2x-3) + 6(x-4)] = (x-4)(2x-$

$3)^2[4x-6+6x-24] = (x-4)(2x-3)^2[10x-30]$, so the critical points are $x = 4, x = 3/2$, and $x = 3$.

(b) $g(x) = (x^2-4)^{2/3}$

Solution: $[f'(x) = 2(x^2-4)^{-1/3} \div 3] \cdot 2x$ so there are critical points of both types, singular and stationary. The stationary critical point is $x = 0$ and the two singular points are the x values that make f' undefined, namely $x = 2$ and $x = -2$.

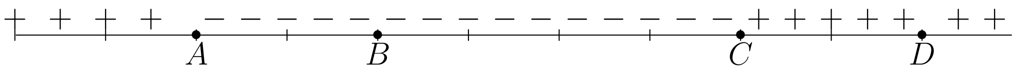
3. (20 points) Given below is a sign chart for the derivative $f'(x)$ of a function.



(a) For each of the stationary points A, B, C and D tell whether $f(x)$ has a relative maximum, relative minimum, or neither at the point.

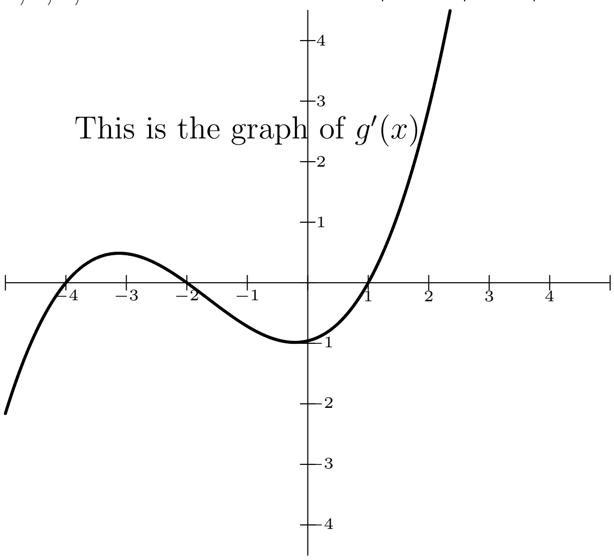
Solution: Since f is increasing to the left of A and decreasing just to the right of A, it must have a local max at A. Since f' has the same sign on both sides of B and D, it has neither a max nor a min at these points. It has a relative minimum at C.

(b) Suppose $f(x)$ is a polynomial function. Sketch a function on the coordinate system below that could have a derivative whose sigh chart is the one given.



Solution: Look for this answer in class after the test.

4. (15 points) Consider the cubic polynomial $p(x)$ whose graph is given. Note the x-intercepts are -4, -2 and 1 and the y-intercept is -1. Find numbers a, b, c, and d such that $ax^3 + bx^2 + cx + d$ has the given graph.



Solution: Since the x-intercepts are $-4, -2$, and 1, the linear factors must be $x+4$, $x+2$, and $x-1$, so the polynomial has the form $p(x) = a(x+4)(x+2)(x-1)$. Since $p(0) = -1 = a(4)(2)(-1) = -8a$, it follows that $a = 1/8$. Therefore $p(x) = (x+4)(x+2)(x-1)/8 = x^3/8 + 5x^2/8 + x/4 - 1$. So a, b, c, and d are $1/8, 5/8, 1/4$, and -1 respectively.

5. (20 points) The altitude of a rocket in feet t seconds into the flight is given by $s = f(t) = -t^3 + 96t^2 + 195t + 5$.

   (a) What is the maximum altitude attained by the rocket? At what time into the flight does this occur?

   Solution: The maximum altitude is attained at some point where the velocity is zero. The velocity function $v(t)$ is given by $v(t) = -3t^2 + 192t + 195$ whose zeros we find by factoring: $t = -1$ and $t = 65$. Now the maximum height is $f(65) = 143655$ feet.

   (b) What is the maximum velocity attained by the rocket? At what time into the flight does this occur?

   Solution: To find the maximum velocity, we need to find the critical points of the velocity function. So compute $v'(t) = -6t + 192$, which has only one zero, $t = 32$. At time $t = 32$, $v(t) = -3 \cdot 32^2 + 192 \cdot 32 + 195 = 3267$.

6. (15 points) Four identical $x \times x$ square corners are cut from a 14×20 inch rectangular piece of metal, and the sides are folded upward to build a box.

   (a) What is the volume of the box that results when the corners cut are 1×1.

   Solution: The volume is the product length $\times$ width $\times$ height $= 12 \times 16 \times 1 = 216$.

   (b) Let $V(x)$ denote the volume of the box when the $x \times x$ corners are removed. Find $V(2)$ and $V(3)$.

   Solution: Note that $V(x) = (14-2x)(20-2x)(x)$ and $V(2) = 10 \cdot 16 \cdot 2 = 320$ and $V(3) = 8 \cdot 14 \cdot 3 = 336$.

   (c) What is the implied domain of V?

   Solution: The domain of V is $[0, 7]$.

   (d) Find $V'(x)$.

   Solution: $V'(x) = \frac{d}{dx}(14x - 2x^2)(20 - 2x) = (14 - 4x)(20 - 2x) - (14x - 2x^2)(-2) = 12x^2 - 136x + 280$.

   (e) Find the critical points of $V(x)$.

   Solution: Factor out 4 from $V'(x)$ and use the quadratic formula to find the zeros of $3x^2 - 34x + 70$. We get

   $$x = \frac{34 \pm \sqrt{34^2 - 4 \cdot 3 \cdot 70}}{6} = \frac{34 \pm \sqrt{316}}{6} = 2.7039$$

   when we take the negative sign. The other place where $V'(x) = 0$ is outside the interval $[0, 7]$. Its about 8.629.

   (f) What value of x makes the value of V maximum? Estimate within .01 the maximum value of V.

   Solution: $V(2.7039) \approx 339.01255 \approx 339.01$.

7. (20 points) Amber Airlines runs chartered flights to Costa Rica. They expect 200 passengers and they charge each passenger \$300. However if more than 200 persons sign up for the flight, they agree to charge \$0.75 less per ticket for each extra person.

   (a) Find the revenue function $R(x)$ in terms of the number of new passengers x. In other words, let $x + 200$ represent the number of passengers, where $x > 0$

   Solution: Let x represent the number of passengers beyond 200 that Amber Airlines enlists. Then $R(x) = (200 + x)(300 - 0.75x)$.

   (b) How many passengers result in the maximum revenue?

   Solution: To maximize $R(x)$, find the critical points and the endpoints of the domain. The domain is $[0, \infty)$, and, by the product rule, the derivative is $R'(x) = 1(300-.75)-0.75(200+x) = 300-.75x-150-.75x = 150-1.5x$. So $x = 100$ is the only critical point. Note that $R''(x) = -1.5$, so $R''(100) = -1.5 < 0$, and this means that $x = 100$ is the location of a relative maximum. Since $R(0) = 200 \cdot 300 = 60000$ is the only endpoint, and since R is decreasing to the right of $x = 100$ (why?, $R'(x)$ is negative for $x > 100$), it follows that R has an absolute maximum at $x = 100$.

   (c) What is that maximum revenue?

   Solution: The maximum revenue is $R(100) = 300 \cdot 225 = 67500$.