August 2, 2005 **Name** ____________________

The total number of points available is 160. Throughout this test, **show your work.**

1. (10 points) Consider the function $f(x) = 5x^2 - 8x + 3, \quad 0 \leq x \leq 7$. Find the locations of the absolute maximum of $f(x)$ and the absolute minimum of $f(x)$ and the value of f at these points.

   Solution: Since $f'(x) = 10-8$ we have one critical point at $x = 0.8$. The other two candidates for extrema are the endpoints, 0 and 7. Checking functional values, we have $f(0) = 3$, $f(0.8) = -0.2$, and $f(7) = 192$. So f has an absolute maximum of 192 at $x = 7$ and an absolute minimum of -0.2 at $x = 0.8$.

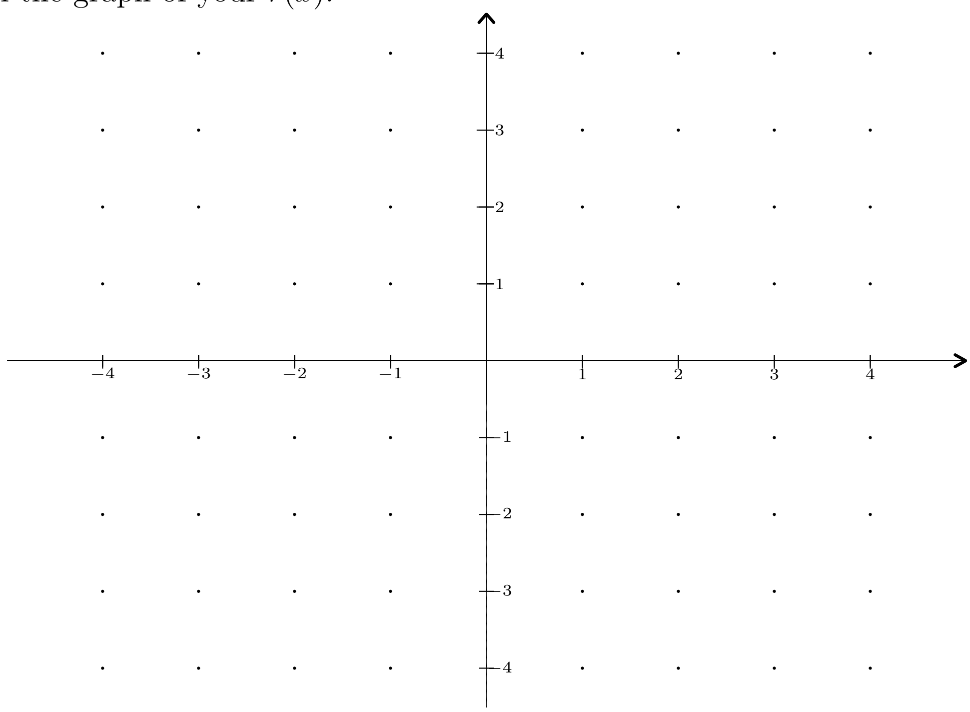
2. (20 points) Find a rational function $r(x)$ that has all the following properties:

   (a) It has exactly two zeros, $x = -2$ and $x = 3$.

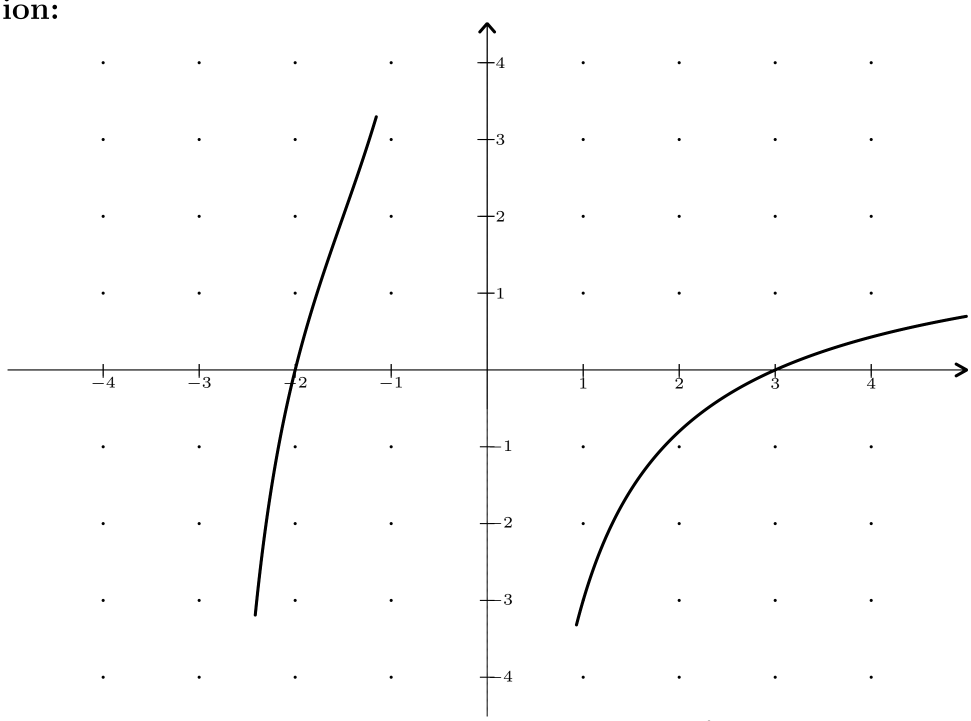
   (b) It has two vertical asymptotes, $x = 0$ and $x = -3$.

   (c) It has $y = 2$ as a horizontal asymptote.

   (a) Sketch the graph of your $r(x)$.



Solution:



Sadly, you cannot see the little curly part in the upper left corner.

(b) Find a symbolic representation of r.

Solution: There are a few ways to do this. The easiest is to make the numerator $2(x+2)(x-3)$ and the denominator $x(x+3)$. The graph is shown above.

3. (50 points) Consider the function $f(x) = (2x-1)^2(x+3)^2$.

(a) Find $f'(x)$ and $f''(x)$.

Solution: By the product rule, $f'(x) = 2(2x-1)(2(x+3)^2 + 2(x+3)(2x-1)^2 = 2(2x-1)(x+3)(2(x+3)+2x-1) = 2(2x-1)(x+3)(4x+5)$. and $f''(x) = 2[2(x+3)(4x+5)+(2x-1)(4x+5)+4(2x-1)(x+3)] = 2[24x^2+60x+13]$.

(b) Find the three critical points of f.

Solution: The zeros of f' are $-3, -1.25$, and 0.5. There are no singular points.

(c) Apply the Test Interval Technique to find the sign chart for f' and use the information in the sign chart to classify the critical points of f. In other words, tell whether each one is the location of (a) a relative maximum, (b) a relative minimum, or (c) neither a relative max or min.

Solution: From the work above, we see that the intervals are $(-\infty, -3)$, $(-3, -1.25), (-1.25, 0.5), (0.5, \infty)$. The sign of f' is negative over the first and third of these and positive over the other two.

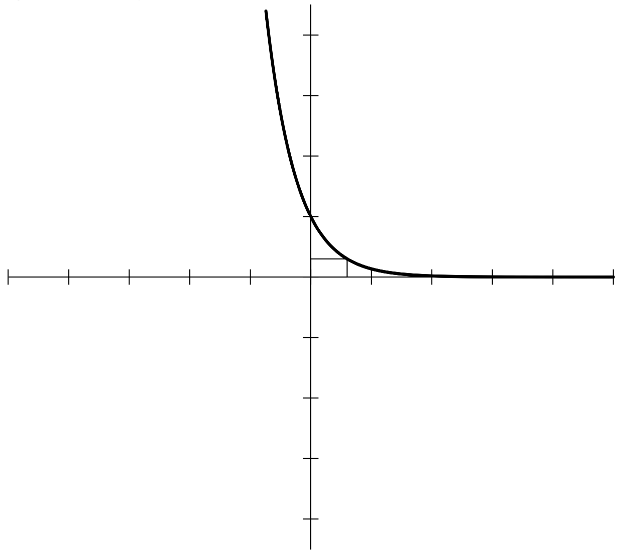
(d) List the intervals over which f is increasing.

Solution: Reading from the sign chart for f', we see that f is increasing over $(-3, -1.25)$ and $(0.5, \infty)$.

(e) Discuss the concavity to f and find all the inflection points on the graph of f.

Solution: Use the quadratic formula to solve $24x^2 + 60x + 13 = 0$ to get the two roots $x \approx -2.26$ and $x \approx -0.24$. Since f'' is a parabola that opens upward, we can see that $f''(x) < 0$ between the two roots and positive outside the two. So f is concave upwards on $(\infty, -2.26)$ and on $(-0.24, \infty)$ and concave downward on the interval between them. There are points of inflection at roughly $(-0.24, f(-0.24))$ and $(-2.26, f(-2.26))$.

4. (20 points) Consider the function $g(x) = e^{-2x}$. A rectangle R with sides parallel to the x- and y-axes has its lower left vertex at the origin and its upper right vertex on the graph of g as shown below.



(a) Note that the area of R depends only on the choice of x. Find the area $R(x)$. For example, $R(2) = 2 \cdot e^{-4}$.

Solution: The area function is $R(x) = xe^{-2x}, 0 \leq x$. Thus, $R'(x) = e^{-2x} - 2xe^{-2x}$, by the product rule. The only critical point is $x = 1/2$.

(b) Find the value of x that maximizes the area of the rectangle. What is it about the sign chart of $R'(x)$ that convinces you the you have found a relative maximum.

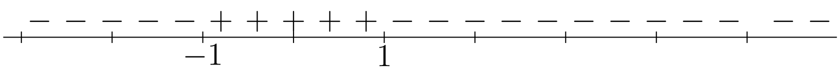
Solution: The sign chart for $R'(x)$ makes it clear that a maximum is realized at $x = 1/2$. The sign chart for $R'(x)$ is positive to the left of $1/2$ and negative to the right of $1/2$.

5. (20 points) Consider the function $f(x) = \ln(x^2 + 1)$.

   (a) Find $f'(x)$.
   Solution: $f'(x) = \frac{2x}{x^2+1}$.

   (b) Find $f''(x)$.
   Solution: $f''(x) = \frac{2(x^2+1)-2x(2x)}{(x^2+1)^2}$.

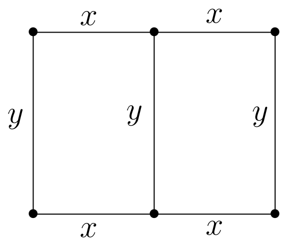
   (c) Find the sign chart for $f''(x)$.
   Solution: $f''(x) < 0$ on $(-\infty, -1)$ and on $(1, \infty)$ and positive on $(-1, 1)$, as shown on the sign chart for f'':



   (d) Find the intervals over which f is concave upwards.
   Solution: From (c) it follows that f is concave upwards on $(-1, 1)$.

6. (20 points) A rancher wants to fence in an area of 10 square miles in a rectangular field and then divide it in half with a fence down the middle parallel to one side. What is the shortest length of fence that the rancher can use?

   Solution: About 16 miles of fencing is needed. See the diagram below. Note that the total amount of fencing needed, based on the labeling of the figure is $3y + 4x$ and the area fenced in is $A = 10 = 2xy$. Solve the last relation for y to get $y = 5/x$. Now the amount of fencing f can be written in terms of x as follows: $f(x) = 3(5/x) + 4x, 0 < x$. Find the critical points of f by first noting that $f'(x) = 15(-1)x^{-2} + 4$. Then solve $f'(x) = 0$ to get $x = \sqrt{15}/2$. The sign chart for f' shows that f has a minimum at $\sqrt{15}/2$. The rancher needs $f(\sqrt{15}/2) = 4\sqrt{15} \approx 15.49$ miles of fencing.



7. (20 points) A baseball team plays in he stadium that holds 60000 spectators. With the ticket price at 12 dollars the average attendance has been 25000. When the price dropped to 10 dollars, the average attendance rose to 40000.

   (a) Find the demand function $p(x)$, where x is the number of the spectators and $p(x)$ is measured in dollars, assuming it is linear. In other words, if the relationship between the price and number of tickets sold is linear, find the price when x tickets are sold.

   Solution: We need to find the linear demand function, given that $(25000, 12)$ and $(40000, 10)$ are on the graph. To simplify, we measure the attendance in thousands, so the two points are $(25, 12)$ and $(40, 10)$. Thus the slope is $m = \frac{12-10}{25-40} = -\frac{2}{15}$. Using the point-slope form of a line, we have $p(x) - 10 = -2/15(x - 40)$. Simplifying yields $p(x) = (-2x + 230)/15$.

   (b) How should the ticket price be set to maximize revenue?

   Solution: Now the revenue function $R(x)$ is the product of number of tickets sold and the price per ticket. Thus $R(x) = xp(x) = x(-2x + 230)/15$. $R'(x) = (-4x+230)/15$, which has a zero at $x = 230/4 = 57.50$. What this says is that the optimum attendance is $57.50 \cdot 1000 = 57500$ and that corresponds to a ticket price of $23/3$ dollars.