

November 21, 2006

Name \_\_\_\_\_

The total number of points available is 139. Throughout this test, **show your work.**

1. (10 points) Find all vertical and horizontal asymptotes, if any, for  $r(x) = \frac{6x^2+3x}{x^2+5x-6}$ .

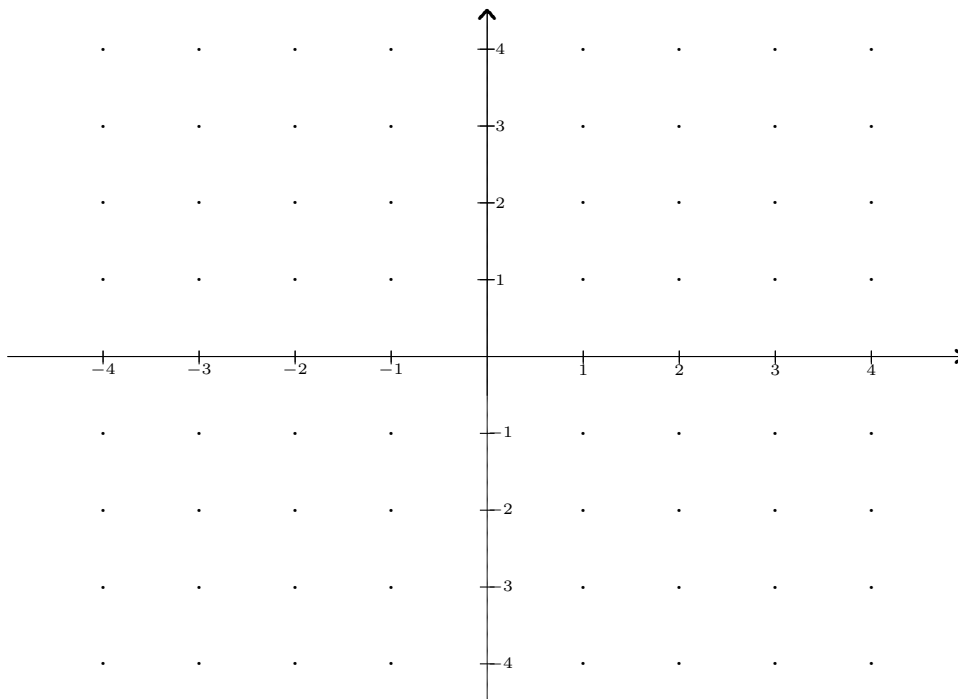
**Solution:** Factor both numerator and denominator to get  $r(x) = \frac{3x(2x+1)}{(x-1)(x+6)}$ , so there are vertical asymptotes at  $x = 1$  and  $x = -6$ , and a horizontal asymptote at  $y = 6$ .

2. (12 points) Consider the function  $f(x) = 4x^3 + 15x^2 - 18x$ ,  $-4 \leq x \leq 4$ . Find the locations of the absolute maximum of  $f(x)$  and the absolute minimum of  $f(x)$  and the value of  $f$  at these points.

**Solution:** Since  $f'(x) = 12x^2 + 30x - 18 = 6(2x^2 + 5x - 3) = 6(2x - 1)(x + 3)$  we have critical points at  $x = 0.5$  and  $x = -3$ . The other two candidates for extrema are the endpoints,  $-4$  and  $4$ . Checking functional values, we have  $f(-4) = 56$ ,  $f(0.5) = -19/4$ ,  $f(-3) = 81$  and  $f(4) = 424$ . So  $f$  has an absolute maximum of 424 at  $x = 4$  and an absolute minimum of  $-19/4$  at  $x = 0.5$ .

3. (12 points) Sketch the graph of a function  $f$  that satisfies the following conditions.

- (a) The domain of  $f$  is  $(-\infty, 1) \cup (1, \infty)$
- (b)  $x = 1$  is a vertical asymptote.
- (c)  $\lim_{x \rightarrow \infty} f(x) = 2, \lim_{x \rightarrow -\infty} f(x) = 0$ .
- (d)  $f(-1) = 1, f(0) = 0$
- (e)  $f'(x) > 0$  on  $(-\infty, -1) \cup (0, 1)$
- (f)  $f'(x) < 0$  on  $(-1, 0) \cup (1, \infty)$
- (g)  $f''(x) > 0$  on  $(-\infty, -2) \cup (-\frac{1}{2}, 1) \cup (1, \infty)$
- (h)  $f''(x) < 0$  on  $(-2, -\frac{1}{2})$



4. (20 points) For each function listed below, find all the critical points. Recall that the derivative of the exponential function  $e^{f(x)}$  is given by  $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$ . It is also useful to know that the equation  $e^x = 0$  has no solutions. Tell whether each critical point gives rise to a local maximum, a local minimum, or neither.

(a)  $f(x) = (x^2 - 4)^3$

**Solution:**  $f'(x) = 3(x^2 - 4)^2 \cdot 2x$ , so the critical points are  $x = \pm 2, x = 0$ . Looking at the sign chart of  $f'$ , we see that  $f'$  does not change signs at  $\pm 2$ , so neither of these is an extremum. But  $f'$  changes from negative to positive at zero, so  $f$  must have a minimum there.

(b)  $g(x) = (x + 2)^{2/3}$

**Solution:**  $g'(x) = \frac{2}{3}(x + 2)^{-1/3}$ , which means that  $g$  has a singular point at  $x = -2$ . Since  $f'$  is negative to the left of  $-1$  and positive to the right, we know  $f$  has a minimum at  $x = -2$ .

(c)  $h(x) = e^{x^3 - 3x}$

**Solution:**  $h'(x) = (3x^2 - 3)(e^{x^3 - 3x})$ , so we solve  $3x^2 - 3 = 0$  to get two critical points,  $x = \pm 1$ . Looking at the sign chart of  $h'$ , we see that  $h'$  is negative between  $-1$  and  $1$ , and positive elsewhere. So  $h$  has a max. at  $-1$  and a min. at  $1$ .

(d)  $k(x) = \frac{x^2 - 1}{2x + 3}$

**Solution:** By the quotient rule,  $k'(x) = \frac{2x(2x+3) - 2(x^2-1)}{(2x+3)^2}$ , which we set equal to zero and solve. Using the quadratic formula, we get  $\frac{-3 \pm \sqrt{9-4}}{2} = -\frac{3}{2} \pm \frac{\sqrt{5}}{2}$ . Looking at the sign chart of  $k'$ , we see that  $k'$  is negative between the two zeros, and positive elsewhere. So  $k$  has a max. at the first one and a min. at the second.

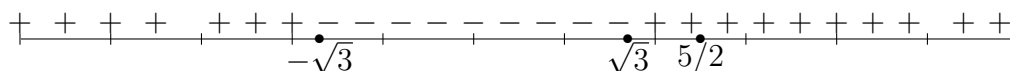
(e)  $f(x) = \ln(x^2 + 4)$

**Solution:**  $f'(x) = \frac{2x}{x^2+4}$ , so there is only one critical point,  $x = 0$ . The sign chart of  $f'$  tells us that  $f$  has a minimum at  $0$ .

5. (25 points) Consider the function  $f(x) = (2x - 5)^2(x^2 - 3)$ .

(a) Find the places where  $f(x)$  changes signs.

**Solution:** The function could change signs at any of  $x = 5/2, x = \sqrt{3}, x = -\sqrt{3}$ , but the sign chart shows that  $f$  does not change signs at  $5/2$ .



(b) Find the places where  $f(x)$  has a horizontal tangent line.

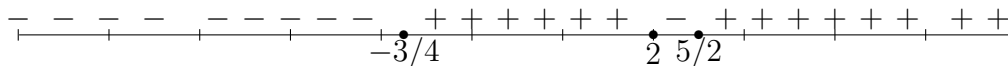
**Solution:** Use the product rule to differentiate  $f$ , getting  $f'(x) = 2(2x - 5)(4x^2 - 5x - 6) = 2(2x - 5)(x - 2)(4x + 3)$ . The zeros of  $f'$  are  $3/2, 2$ , and  $-3/4$ . There are no singular points.

(c) Find the places where  $f(x)$  changes concavity.

**Solution:** Use the product rule to find that  $f''(x) = 2 \cdot 2(4x^2 - 5x - 6) + 2(2x - 5)(8x - 5) = 48x^2 - 120x + 26$ , which has two zeros,  $\alpha = \frac{60 - \sqrt{60^2 - 4 \cdot 24 \cdot 13}}{48} \approx 0.2396$  and  $\beta = \frac{60 + \sqrt{60^2 - 4 \cdot 24 \cdot 13}}{48} \approx 2.260$ , by the quadratic formula.

(d) For the points in part (b), tell whether  $f(x)$  has (a) a relative maximum, (b) a relative minimum, or (c) neither a relative max or min.

**Solution:** The sign chart for  $f'(x)$  is shown below. From it you can deduce that  $f$  is falling to the left of  $-2$ , rising between  $-2$  and  $3/4$ , falling again from  $3/4$  to  $5/2$ , and then increasing to the right of  $5/2$ . It follows that  $f$  has relative mins at  $-3/4$  and  $5/2$  and a relative max at  $2$ .



6. (10 points) Find a rational function  $r(x)$  that has the following properties.

- (a) It has exactly two zeros,  $x = -4$  and  $x = 2$ .
- (b) It has two vertical asymptotes,  $x = 0$  and  $x = -1$ .
- (c) It has  $y = 2$  as a horizontal asymptote.

**Solution:** One function that works is

$$r(x) = \frac{2(x+4)(x-2)}{x(x+1)}.$$

7. (15 points) Find the point on the line  $3x + 2y = 6$  that is closest to the point  $(-4, -4)$ .

**Solution:** The points on the line all satisfy  $(x, y) = (x, 3 - 3x/2)$ , so the distance function we want to minimize is  $D(x) = \sqrt{(x+4)^2 + (3 - 3x/2 + 4)^2} = \sqrt{(x+4)^2 + (7 - 3x/2)^2}$ . We can alternatively minimize  $D^2$ . Let  $H(x) = D^2(x) = (x+4)^2 + (7 - 3x/2)^2$ . Then  $H'(x) = 2(x+4) + 2(7 - 3x/2) \cdot (-3/2)$ . Setting  $h'(x)$  equal to zero to find the critical point(s),  $2x + 8 - 3(7 - 3x/2) = 2x + 8 - 21 + 9x/2 = 13x/2 - 13 = 0$  exactly when  $x = 2$ . Therefore the point on the line closest to  $(-4, -4)$  is  $(2, 0)$  and the minimum distance is  $\sqrt{52}$ .

8. (15 points) A baseball team plays in the stadium that holds 60000 spectators. With the ticket price at 12 dollars the average attendance has been 25000. When the price dropped to 10 dollars, the average attendance rose to 40000.

- (a) Find the demand function  $p(x)$ , where  $x$  is the number of the spectators and  $p(x)$  is measured in dollars, assuming it is linear. In other words, if the relationship between the price and number of tickets sold is linear, find the price when  $x$  tickets are sold.

**Solution:** We need to find the linear demand function, given that (25000, 12) and (40000, 10) are on the graph. To simplify, we measure the attendance in thousands, so the two points are (25, 12) and (40, 10). Thus the slope is  $m = \frac{12-10}{25-40} = -\frac{2}{15}$ . Using the point-slope form of a line, we have  $p(x) - 10 = -2/15(x - 40)$ . Simplifying yields  $p(x) = (-2x + 230)/15$ .

- (b) How should the ticket price be set to maximize revenue?

**Solution:** Now the revenue function  $R(x)$  is the product of number of tickets sold and the price per ticket. Thus  $R(x) = xp(x) = x(-2x + 230)/15$ .  $R'(x) = (-4x + 230)/15$ , which has a zero at  $x = 230/4 = 57.50$ . What this says is that the optimum attendance is  $57.50 \cdot 1000 = 57500$  and that corresponds to a ticket price of  $23/3$  dollars.

9. (20 points) According to Newton's Law of Cooling, the rate at which an object's temperature changes is proportional to the temperature of the medium into which it is emersed. If  $F(t)$  denotes the temperature of a cup of instant coffee (initially  $212^\circ F$ ), then it can be proven that

$$F(t) = T + Ae^{-kt},$$

where  $T$  is the air temperature,  $72^\circ F$ ,  $A$  and  $k$  are constants, and  $t$  is expressed in minutes.

- (a) What is the value of  $A$ ?

**Solution:** Note that  $F(0) = 72 + A \cdot 1 = 212$  so  $A = 140$ .

- (b) Suppose that after exactly 10 minutes, the temperature of the coffee is  $186.6^\circ F$ . What is the value of  $k$ ?

**Solution:** Solve  $F(t) = 186.6 = 72 + 140e^{-k(10)}$  for  $k$  to get  $k = 0.020019$ .

- (c) Use the information in (a) and (b) to find the number of minutes before the coffee reaches the temperature of  $80^\circ F$ .

**Solution:** Solve the equation  $80 = 72 + 140e^{-0.020019t}$  for  $t$  to get first  $e^{-0.020019t} = 8/140 = 0.05714$ , and taking logs of both sides yields  $t = 142.97$  minutes.