April 10, 2007 **Name** ______________________________

The total number of points available is 139. Throughout this test, **show your work.**

1. (10 points) Consider the function $f(x) = \ln(x^2 + 1)$.

   (a) Find the function $f'(x)$.
   Solution: $f'(x) = \frac{2x}{x^2+1}$.

   (b) Find $f'(2)$ and use this number to find an equation for the line tangent to the graph of f at the point $(2, \ln 5)$.
   Solution: $f'(2) = 4/5$ so the line is given by $y - \ln 5 = \frac{4}{5}(x - 2)$. The line very nearly goes through the origin: $y \approx 4x/5 + 0.0094$.

2. (12 points) Consider the function $f(x) = x^3 - 5.5x^2 - 4x + 7, \quad -4 \leq x \leq 6$. Find the locations of the absolute maximum of $f(x)$ and the absolute minimum of $f(x)$ and the value of f at these points.

   Solution: Since $f'(x) = 3x^2 - 11x - 4 = (3x + 1)(x - 4) = 0$ we have the two critical points $x = -1/3$ and $x = 4$. The other two candidates for extrema are the endpoints, -4 and 6. Checking functional values, we have $f(-4) = -129, f(-1/3) \approx 7.685$, $f(6) = 1$ and $f(4) = -33$. So f has an absolute maximum of about 7.685 at $x = -1/3$ and an absolute minimum of -129 at $x = -4$.

3. (30 points including 10 for part (e)) For each function listed below, find all the critical points. Tell whether each critical point gives rise to a local maximum, a local minimum, or neither.

   (a) $f(x) = (x^3 - 8)^2$
   Solution: $f'(x) = 2(x^3 - 8) \cdot 3x^2$, so the critical points are $x = 2$, and $x = 0$. Looking at the sign chart of f', we see that f' does not change signs at $x = 0$, so f does not have an extremum at 0. But f' changes from negative to positive at $x = 2$, so f must have a minimum there.

   (b) $h(x) = e^{2x}/x$
   Solution: $h'(x) = \frac{2xe^{2x} - e^{2x}}{x^2}$, so we set the numerator equal to zero to get $e^{2x}(2x - 1) = 0$ and find a single critical point, $x = \frac{1}{2}$. Looking at the sign chart of h', we see that h' is negative between 0 and 1/2 and to the right of 1/2. So h has a min. at 1/2.

(c) $h(x) = x^2 \ln(x+2)$

Solution: $h'(x) = 2x\ln(x+2) + \frac{x^2}{x+2}$. One critical point we can find quickly is $x = 0$. We can solve this on the graphing calculator to get a critical point of $x \approx -0.695$, which is the location of a relative maximum. There is also a relative minimum at $x = 0$.

(d) $g(x) = (x-1)^{2/3}$

Solution: $g'(x) = \frac{2}{3}(x-1)^{-1/3}$, which means that g has a singular point at $x = 1$. Since f' is negative to the left of 1 and positive to the right, we know f has a minimum at $x = 1$.

(e) $k(x) = (2x-3)^2(x^2-1)$

Solution: By the product rule, $k'(x) = 2(2x-3)\cdot 2(x^2-1) + 2x(2x-3)^2 = 2(2x-3)[2(x^2-1) + x(2x-3)] = 2(2x-3)(4x^2-3x-2)$, which we set equal to zero and solve. Using the quadratic formula, we get $\frac{3\pm\sqrt{9+32}}{8}$. So there are three critical points, $x_1 = \frac{3-\sqrt{41}}{8} \approx -0.425$, $x_2 = \frac{3+\sqrt{41}}{8} \approx 1.175$, and $x_3 = 3/2$, Looking at the sign chart of k', we see that k' is negative to the left of x_1, positive between x_1 and x_2, negative between x_2 and x_3, and positive elsewhere. So k has a max. at x_2 and and minimums at x_1 and x_3.

4. (15 points) The function $f(x)$ has been differentiated twice to get $f''(x) = (x^2-4)(x-5)(x+1)$.

(a) Find the places where $f''(x)$ changes signs.

Solution: The function changes signs at $x = \pm 2, x = 5, x = -1$.

(b) Find the places where $f'(x)$ has a horizontal tangent line.

Solution: Again we get the points $x = \pm 2, x = 5, x = -1$.

(c) Find the intervals over which $f(x)$ is concave upwards.

Solution: The function f is concave upwards precisely where f'' is positive. By the test interval technique, this is $(-\infty, -2)$, $(-1, 2)$, and $(5, \infty)$.

5. (15 points) Note to the class. There was a typo (% instead of \$) on the test, so the solution to the problem as stated is quite different from the one given here. When a management training company prices its seminar on management techniques at \$400 per person, 1000 people will attend the seminar. The company estimates that for each \$5 reduction in the price, an additional 20 will attend the seminar. How much should the company charge for the seminar in order to maximize its revenue? What is the maximum revenue?

Solution: Let x represent the number of \$5 price reductions. Then the price per customer is $400 - 5x$ and the number of customers is $1000 + 20x$, so the revenue is given by $R(x) = (400 - 5x)(1000 + 20x)$. We assume that $x \geq 0$ and that $x \leq 80$ since otherwise the cost of the seminar would be negative. Differentiating $R(x)$ and finding critical points yields $x = 15$. Also, $R(15) = \$422,500$ is the absolute maximum value of $R(x)$ over the interval $[0, 80]$. So the company should charge \$325 per person for the seminar.

6. (15 points) It is known from past experiments that the height (in feet) after t months is given by

$$H(t) = 4t^{1/2} - 2t, \quad 0 \leq t \leq 2.$$

How long, on average, does it take a plant to reach its maximum height? What is the maximum height?

Solution: Since $h'(t) = \frac{1}{2}(4t^{-1/2} - 2$, we get $t = 1$ as the only critical point. $H'(t)$ goes from positive to negative at 1, so $H(1) = 2$ is a maximum.

7. (15 points) For a particular person learning to type, it was found that the number N of words per minute the person was able to type after t hours of practice, was given by

$$N = N(t) = 100(1 - e^{-0.02t}).$$

(a) After 10 hours of practice how many words per minute could the person type?

Solution: $N(10) = 100(1 - e^{-0.02(10)} \approx 18.12$.

(b) What was the rate of improvement after 10 hours of practice?

Solution: $N'(t) = 100(0 + .02e^{-0.02t} = 2e^{-0.02t}$ so $N'(10) \approx 1.637$ words per minute per hour of practice time.

(c) What was the rate of improvement after 40 hours of practice?

Solution: $N'(40) = 2e^{-0.8} \approx 0.898$ words per minute per hour.