

November 20, 2007

Name _____

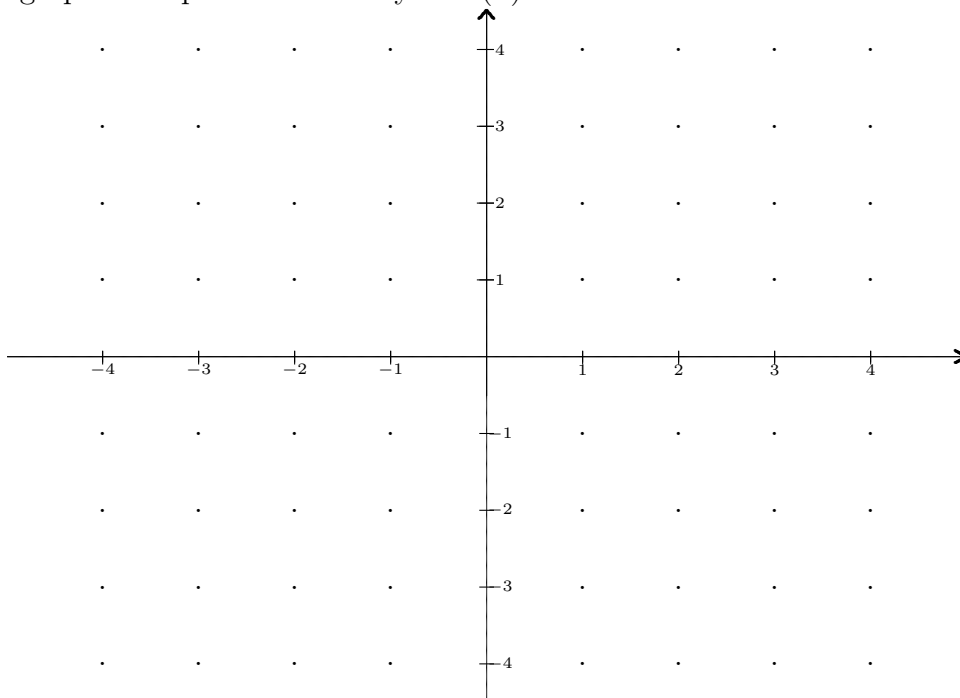
The total number of points available is 134. Throughout this test, **show your work.**

1. (15 points) Consider the function $f(x) = 1 + 9x + 3x^2 - x^3$, $-2 \leq x \leq 6$. Find the locations of the absolute maximum of $f(x)$ and the absolute minimum of $f(x)$ and the value of f at these points.

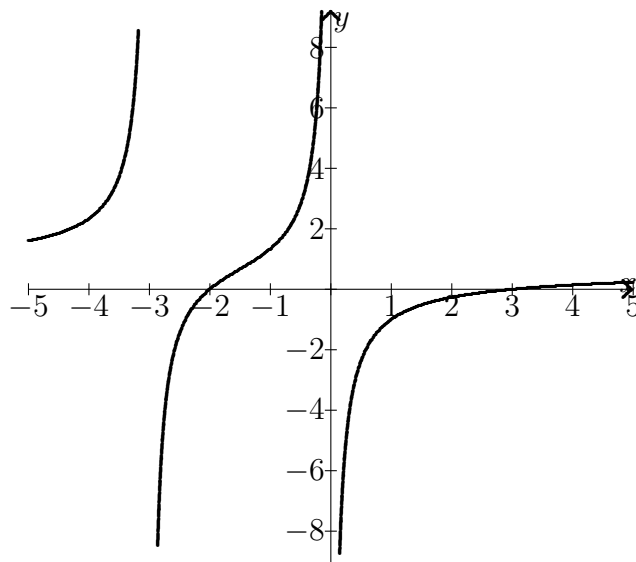
Solution: Since $f'(x) = 9 + 6x - 3x^2 = 3(3 + 2x - x^2) = 3(3 - x)(1 + x)$ we have one critical points at $x = -1$ and $x = 3$. The other two candidates for extrema are the endpoints, -2 and 6 . Checking functional values, we have $f(3) = 28$, $f(-1) = -4$, $f(-2) = 3$ and $f(6) = -53$. So f has an absolute maximum of 28 at $x = 3$ and an absolute minimum of -53 at $x = 6$.

2. (25 points) Find a symbolic rational function $r(x)$ that has all the following properties:
- It has exactly two zeros, $x = -2$ and $x = 3$.
 - It has two vertical asymptotes, $x = 0$ and $x = -3$.
 - It has $y = 2$ as a horizontal asymptote.

(b) Find a graphical representation of your $r(x)$.



Solution:

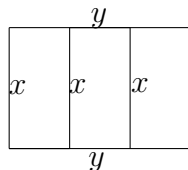


- (a) Find a symbolic representation of r .

Solution: There are a few ways to do this. The easiest is to make the numerator $2(x+2)(x-3)$ and the denominator $x(x+3)$. The graph is shown above.

3. (15 points) A rancher wants to fence in an area of 20 square miles in a rectangular field and then divide it into three smaller rectangular fields using two segments of fence parallel to one side. What is the shortest length of fence that the rancher can use?

Solution: About 25.3 miles of fencing is needed. See the diagram below. Note that the total amount of fencing needed, based on the labeling of the figure is $2y + 4x$ and the area fenced in is $A = 20 = xy$. Solve the last relation for y to get $y = 20/x$. Now the amount of fencing f can be written in terms of x as follows: $f(x) = 2(20/x) + 4x = 40x^{-1} + 4x, 0 < x$. Find the critical points of f by first noting that $f'(x) = 40(-1)x^{-2} + 4$. Then solve $f'(x) = 0$ to get $x = \sqrt{10}$. The sign chart for f' shows that f has a minimum at $x = \sqrt{10}$. The rancher needs $f(\sqrt{10}) = 8\sqrt{10} \approx 25.298$ miles of fencing.



4. (25 points) Consider the function $f(x) = \ln(3x^2 + 1)$.

(a) Find $f'(x)$.

Solution: $f'(x) = \frac{6x}{3x^2+1}$.

(b) Find an equation for the line tangent to the graph of f at the point $(3, f(3))$.

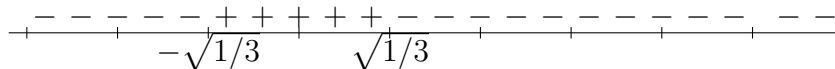
Solution: Since $f'(3) = 18/28 = 9/14$ and $f(3) = \ln 28$, we have $y - \ln 28 = 9(x - 3)/14$.

(c) Find $f''(x)$.

Solution: $f''(x) = \frac{6(3x^2+1) - 6x(6x)}{(3x^2+1)^2}$.

(d) Find the sign chart for $f''(x)$.

Solution: $f''(x) < 0$ on $(-\infty, -\sqrt{1/3})$ and on $(\sqrt{1/3}, \infty)$ and positive on $(-\sqrt{1/3}, \sqrt{1/3})$, as shown on the sign chart for f'' :



(e) Find the intervals over which f is concave upwards.

Solution: From (c) it follows that f is concave upwards on $(-\sqrt{1/3}, \sqrt{1/3})$.

5. (10 points) Find the point on the line $y = 3$ that is closest to the point $(4, -3)$. What is that shortest distance?

Solution: The line is horizontal so the closest point is the one directly above $(4, -3)$ which is $(4, 3)$, which is 6 units away from the given point.

6. (20 points) Four identical $x \times x$ square corners are cut from a 10×10 inch rectangular piece of metal, and the sides are folded upward to build a box.

- (a) What is the volume of the box that results when the corners cut are 1×1 .

Solution: The volume is the product length \times width \times height $= 8 \times 8 \times 1 = 64$.

- (b) Let $V(x)$ denote the volume of the box when the $x \times x$ corners are removed. Find $V(2)$ and $V(3)$.

Solution: Note that $V(x) = (10-2x)(10-2x)(x)$ and $V(2) = 6 \cdot 6 \cdot 2 = 72$ and $V(3) = 4 \cdot 4 \cdot 3 = 48$.

- (c) What is the implied domain of V ?

Solution: The domain of V is $[0, 5]$.

- (d) Find $V'(x)$.

Solution: $V'(x) = \frac{d}{dx}(10-2x)(10x-2x^2) = -2(10x-2x^2) + (10-4x)(10-2x) = 4(3x^2-20x+25)$.

- (e) Find the critical points of $V(x)$.

Solution: Factor the expression to get $x = \frac{5}{3}$ and $x = 5$, both of which belong to the domain. This number, $x = 5/3$.

- (f) What value of x makes the value of V maximum?

Solution: $V(5/3) = 100/9 = 11.\bar{1}$.

7. (12 points) Compound Interest.

- (a) Consider the equation $2000(1 + 0.03)^{4t} = 6000$. Find the value of t and interpret your answer in the language of compound interest.

Solution: t is the time required for an investment at rate $r = 12\%$ compounded quarterly to triple. Use logs to get $t = 9.29$ years.

- (b) Consider the equation $P(1 + 0.04)^{4 \cdot 10} = 5000$. Solve for P and interpret your answer in the language of compound interest.

Solution: P is the principle in dollars required to grow a 16% investment compounded quarterly over 10 years to grow to \$5000. Another way to say this is that P is the present value of \$5000 compounded quarterly over 10 years. Solve the equation to get $P = \$1041.45$

- (c) Consider the equation $Pe^{10r} = 2P$. Solve for r and interpret your answer in the language of compound interest.

Solution: We're compounding continuously, and getting twice the original investment. If we interpret the r as rate, we're asking what rate of interest will cause a continuously compounded 10-year investment to double. Solve $10r = \ln 2$ to get $r = 0.069$ or 6.9%.

8. (12 points) Find an equation for the line tangent to the graph of $f(x) = xe^{-2x+4}$ at the point $(2, f(2))$.

Solution: Find f' first. Then note that $f'(2) = 1 + 2(-2) \cdot 1 = -3$ and $f(2) = 2$, so the line is $y = -3x + 8$.