**April 10, 2008** **Name** ______________________________

The total number of points available is 141. Throughout this test, **show your work.**

1. (15 points) Consider the function $f(x) = 2x^3 - 8x^2 + 3, \quad -2 \leq x \leq 10$. Find the locations of the absolute maximum of $f(x)$ and the absolute minimum of $f(x)$ and the value of $f$ at these points.

   **Solution:** Since $f'(x) = 6x^2 - 16x$ we have one critical point at $x = 0$ and the other at $x = \frac{8}{3}$. The other two candidates for extrema are the endpoints, -2 and 10. Checking functional values, we have $f(0) = 3, f(8/3) \approx -34.92, f(-2) = -45$ and $f(10) = 1203$. So $f$ has an absolute maximum of 1203 at $x = 10$ and an absolute minimum of $-45$ at $x = -2$.

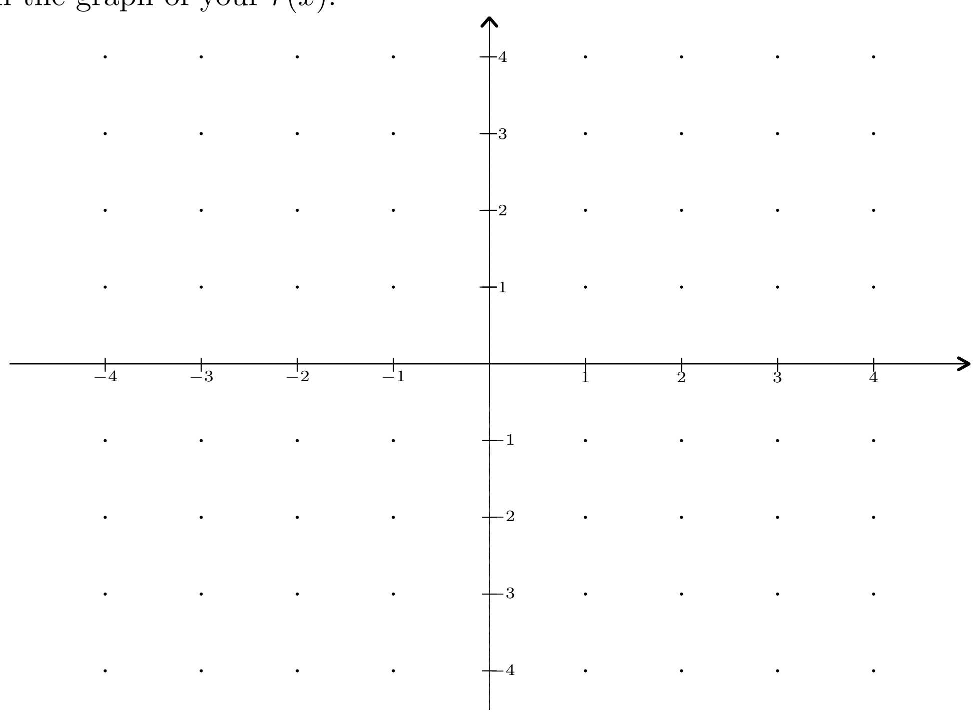
2. (20 points) Find a rational function $r(x)$ that has all the following properties:

   (a) It has exactly two zeros, $x = -3$ and $x = 3$.

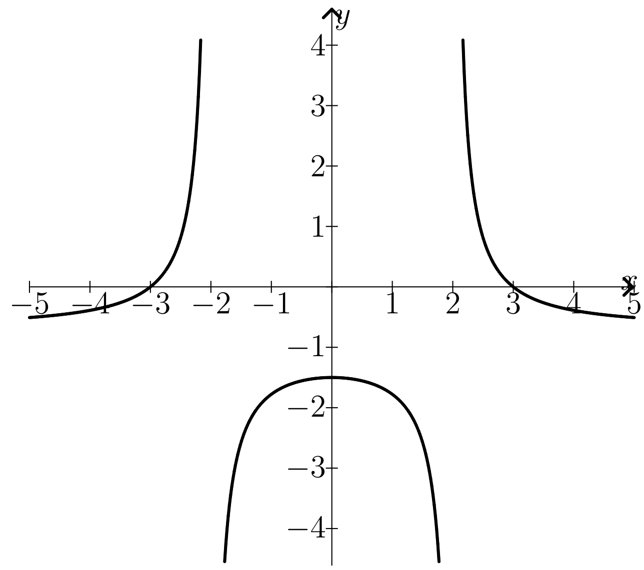
   (b) It has two vertical asymptotes, $x = -2$ and $x = 2$.

   (c) It has $y = -2$ as a horizontal asymptote.

   (a) Sketch the graph of your $r(x)$.



**Solution:**




(b) Find a symbolic representation of $r$.

**Solution:** There are a few ways to do this. The easiest is to make the numerator $2(x+3)(x-3)$ and the denominator $(x-2)(x+2)$. To make the function have $y=-2$ as an asymptote, we can simply multiply by $-2$. Thus $r(x) = \frac{-2x^2+18}{x^2-4}$ .

3. (30 points) Consider the function $f(x) = (2x + 3)^2(x - 4)^2$.

   (a) Find $f'(x)$ and $f''(x)$.

   **Solution:** By the product rule, $f'(x) = 2(2x + 3)(x - 4)(4x - 5)$. and $f''(x) = 2(24x^2 - 60x - 23)$.

   (b) Find all the critical points of $f$.

   **Solution:** The zeros of $f'$ are $-3/2, 1.25$, and 4. There are no singular points.

   (c) Apply the Test Interval Technique to find the sign chart for $f'$ and use the information in the sign chart to classify the critical points of $f$. In other words, tell whether each one is the location of (a) a relative maximum, (b) a relative minimum, or (c) neither a relative max or min.

   **Solution:** From the work above, we see that the intervals are $(-\infty, -3/2), (-3/2, 1.25), (1.25, 4), (4, \infty)$. The sign of $f'$ is negative over the first and third of these and positive over the other two.

   (d) List the intervals over which $f$ is increasing.

   **Solution:** Reading from the sign chart for $f'$, we see that $f$ is increasing over $(-3/2, 1.25)$ and $(4, \infty)$.

   (e) Discuss the concavity to $f$ and find all the inflection points on the graph of $f$.

   **Solution:** Use the quadratic formula to solve $24x^2 - 60x - 23 = 0$ to get the two roots $x \approx -0.3377$ and $x \approx 2.8377$. Since $f''$ is a parabola that opens upward, we can see that $f''(x) < 0$ between the two roots and positive outside the two. So $f$ is concave upwards on $(\infty, -0.3377)$ and on $(2.8377, \infty)$ and concave downward on the interval between them. There are points of inflection at roughly $(-0.3377, f(-0.3377))$ and $(2.83, f(2.83))$.

4. (15 points) Find an equation for the line tangent to (the graph of) $f(x) = xe^{2x}$ at the point $(2, 2e^4)$.

   **Solution:** Since $f' = e^{2x} + 2xe^{2x}$, by the product rule, it follows that $f'(2) = e^4 + 4e^4 = 5e^4$, and that the tangent line is given by $y - 2e^4 = 5e^4(x - 2)$, which in slope-intercept form is $y = 5e^4 x - 8e^4$.

5. (20 points) Certain radioactive material decays in such a way that the mass remaining after $t$ years is given by the function

$$m(t) = 165e^{-0.02t}$$

   where $m(t)$ is measured in grams.

   (a) Find the mass at time $t = 0$.

   **Solution:** $m(0) = 165$ grams.

   (b) How much of the mass remains after 15 years?

   **Solution:** $m(15) = 122.23$ grams.

   (c) What is the half-life of the material?

   **Solution:** Solve the equation $e^{-0.02t} = \frac{82.5}{165} = \frac{1}{2}$ which leads to $-0.02t = \ln(0.5)$ for which the value of $t$ is about 34.657 years.

   (d) Find the rate of loss at $t = 1$ year.

   **Solution:** Since $m'(t) = 165(-0.02)e^{-0.02t}$, it follows that $m'(1) \approx -3.234$ grams per year.

6. (10 points) The population of the world in 1990 was 5 billion and the relative growth rate was estimated at 1.5 percent per year. Assuming that the world population follows an exponential growth model, find the projected world population in 2010.

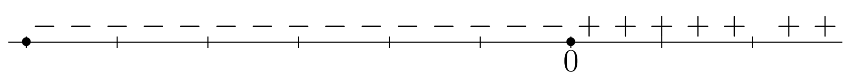
   **Solution:** The function is $P(t) = p_0 e^{rt}$, and we know $P(0) = 5$ billion, so $P(20) = 5 \cdot e^{0.15(20)} = 5e^{0.3} \approx 6.749$ billion.

7. (20 points) Consider the function $f(x) = x\ln(x^2 + 2)$.

   (a) Find $f'(x)$.
   **Solution:** $f'(x) = \ln(x^2 + 2) + \frac{2x^2}{x^2+2}$.

   (b) Find $f''(x)$.
   **Solution:** $f''(x) = \frac{2x}{x^2+2} + \frac{4x(x^2+2)-2x(2x^2)}{(x^2+2)^2} = \frac{4x}{x^2+2}$ after a lot of arithmetic.

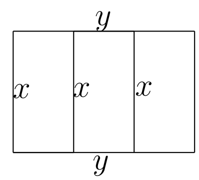
   (c) Find the sign chart for $f''(x)$.
   **Solution:** $f''(x) < 0$ on $(-\infty, 0)$ and positive on $(0, \infty)$, as shown on the sign chart for $f''$:




   (d) Find the intervals over which $f$ is concave upwards.
   **Solution:** From (c) it follows that $f$ is concave upwards on $(0, \infty)$.

8. (12 points) A rancher wants to fence in an area of 12 square miles in a rectangular field and then divide it in half with a fence down the middle parallel to one side. What is the shortest length of fence that the rancher can use?

   **Solution:** About 17 miles of fencing is needed. See the diagram below. Note that the total amount of fencing needed, based on the labeling of the figure is $2y + 3x$ and the area fenced in is $A = 12 = xy$. Solve the last relation for $y$ to get $y = 12/x$. Now the amount of fencing $f$ can be written in terms of $x$ as follows: $f(x) = 2(12/x) + 3x, 0 < x$. Find the critical points of $f$ by first noting that $f'(x) = 3 - 24x^{-2}$. Then solve $f'(x) = 0$ to get $x = \sqrt{8}$. The sign chart for $f'$ shows that $f$ has a minimum at $\sqrt{8}$. The rancher needs $f(\sqrt{8}) = 12\sqrt{2} \approx 16.97$ miles of fencing.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