November 23, 2010 **Name** ______________________________

The total number of points available is 153. Throughout this test, **show your work. Throughout this test, you are expected to use calculus to solve problems. Graphing calculator solutions will generally be worth substantially less credit.**

1. (12 points) Find an equation for the line tangent to the graph of $f(x) = xe^{-2x+4}$ at the point $(2, f(2))$.

   Solution: Find f' first. Then note that $f'(2) = 1 + 2(-2) \cdot 1 = -3$ and $f(2) = 2$, so the line is $y = -3x + 8$.

2. (12 points) Find an equation for the line tangent to the graph of $f(x) = x^2 \ln(x)$ at the point (e, e^2).

   Solution: First $f'(x) = 2x \ln(x) + x^2/x$. Then note that $f'(e) = 2e + e = 3e$, so the line is $y - e^2 = 3e(x - e)$.

3. (12 points) A radioactive substance has a half-life of 27 years. Find an expression for the amount of the substance at time t if 20 grams were present initially.

   Solution: $Q(t) = Q_0 e^{-kt}$. Since $Q_0 = 20$ and the half-life is 27 years, it follows that $10 = 20e^{-27k}$, which can be solved to give $k \approx 0.025672$. Thus $Q(t) = 20e^{-0.025672t}$.

4. (15 points) For each function f listed below, find the slope of the line tangent to its graph at the point $(0, f(0))$.

   (a) $f(x) = e^{e^x}$.
   Solution: $f'(x) = e^{e^x} \cdot e^x$, so $f'(0) = e^{e^0} \cdot e^0 = e$.

   (b) $f(x) = (x-1)^2 \cdot \ln(2x+1)$.
   Solution: $f'(x) = 2(x-1) \cdot \ln(2x+1) + \frac{2}{2x+1}(x-1)^2$, so $f'(0) = 2(0-1) \cdot \ln(1) + \frac{2}{1}(0-1)^2 = 2$.

   (c) $f(x) = (1 + \ln(2x+1))^3$.
   Solution: $f'(x) = 3(1+\ln(2x+1))^2 \cdot (0 + \frac{2}{2x+1}) = 3(1+0)^2(2) = 6$.

5. (10 points) For each function listed below, find a critical point.

   (a) $g(x) = x\ln(x)$.
   Solution: $g'(x) = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$. Then $\ln(x) = -1$ when $x = e^{-1}$.

   (b) $h(x) = (2x-3)e^{4x}$.
   Solution: $h'(x) = 2e^{4x} + 4e^{4x}(2x-3) = e^{4x}(2 + 8x - 12)$. Therefore, we have $8x - 10 = 0$ and $x = 1.25$.

6. (15 points) Consider the function $f(x) = 1+9x+3x^2-x^3, \quad -2 \le x \le 6$. Find the locations of the absolute maximum of $f(x)$ and the absolute minimum of $f(x)$ and the value of f at these points.

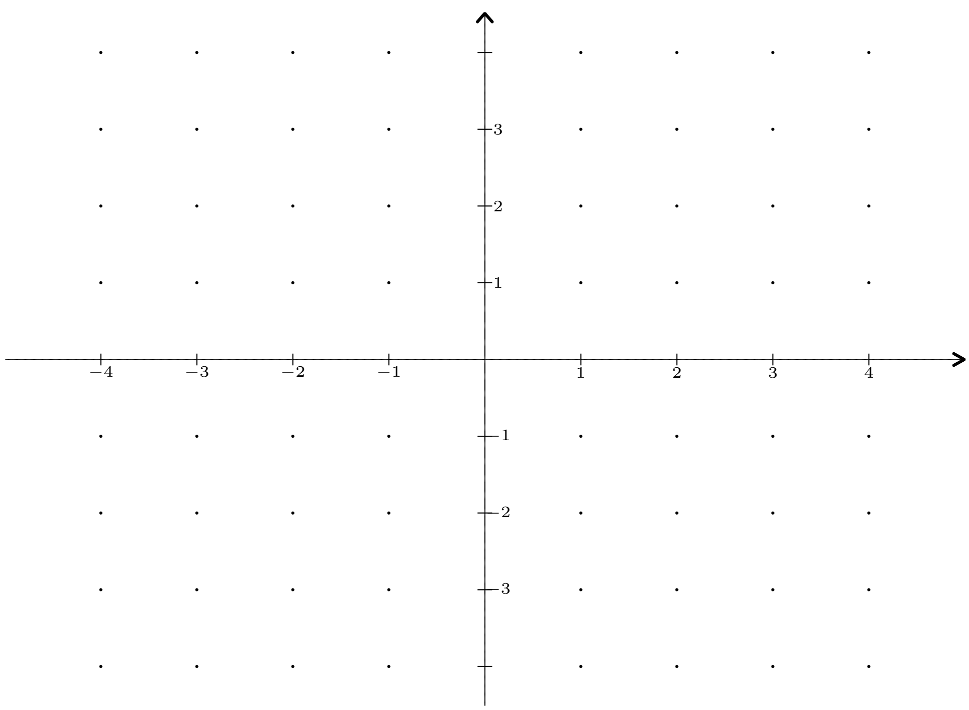
   Solution: Since $f'(x) = 9 + 6x - 3x^2 = 3(3 + 2x - x^2) = 3(3 - x)(1 + x)$ we have one critical points at $x = -1$ and $x = 3$. The other two candidates for extrema are the endpoints, -2 and 6. Checking functional values, we have $f(3) = 28, f(-1) = -4$, $f(-2) = 3$ and $f(6) = -53$. So f has an absolute maximum of 28 at $x = 3$ and an absolute minimum of -53 at $x = 6$.

7. (15 points) Consider the function $f(x) = \frac{(2x+3)(x-3)}{x(x-1)}$.

   (a) Build the sign chart for f
   Solution: We have to use all the points where f could change signs, $x = -3/2, 3, 0$, and 1. As expected the signs alternate starting with + at the far left:$+ - + - +$.

   (b) Find the vertical and horizontal asymptotes.
   Solution: The vertical asymptotes are $x = 0$ and $x = 1$, and the horizontal asymptote is $y = 2$.

   (c) Use the information from the first two parts to sketch the graph of f.



8. (25 points) According to Newton's Law of Cooling, the rate at which an object's temperature changes is proportional to the temperature of the medium into which it is emersed. If $F(t)$ denotes the temperature of a cup of instant coffee (initially $212°F$), then it can be proven that

$$F(t) = T + Ae^{-kt},$$

where T is the air temperature, $62°F$, A and k are positive constants, and t is expressed in minutes.

(a) What is the value of A?

Solution: Since $F(t) = 62+Ae^{-kt}$, it follows that $F(0) = 62+A\cdot 1 = 212$ so $A = 150$.

(b) Suppose that after exactly 14 minutes, the temperature of the coffee is $112°F$. What is the value of k, correct to four places?

Solution: Solve $F(14) = 112 = 62+150e^{-k(14)}$ for k to get $k = \ln 3 \div 14 \approx 0.0784$.

(c) Use the information in (a) and (b) to find the number of minutes before the coffee reaches the temperature of $80°F$.

Solution: Solve the equation $80 = 62 + 150e^{-0.0784t}$ for t to get first $e^{-0.0784t} = 18/150 = 0.12$, and taking logs of both sides yields $t = 27.0$ minutes.

(d) Take the derivative of $F(t)$ and show that the rate of change of temperature of the coffee is proportional to the difference between the room temperature and the coffee's temperature. That is, show that $F'(t) = l(F(t) - T)$ where l is a constant of proportionality. This is actually much easier than it might seem.

Solution: $F'(t) = 0+A(-k)e^{-kt}$ while $F(t)-T = T+Ae^{-kt}-T = Ae^{-kt}$, so the constant l is $-k$

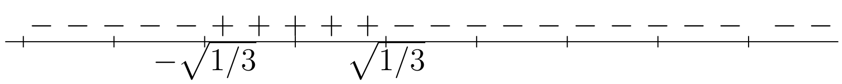
9. (25 points) Consider the function $f(x) = \ln(3x^2 + 1)$.

   (a) Find $f'(x)$.
   Solution: $f'(x) = \frac{6x}{3x^2+1}$.

   (b) Find an equation for the line tangent to the graph of f at the point $(3, f(3))$.
   Solution: Since $f'(3) = 18/28 = 9/14$ and $f(3) = \ln 28$, we have $y - \ln 28 = 9(x-3)/14$.

   (c) Find $f''(x)$.
   Solution: $f''(x) = \frac{6(3x^2+1)-6x(6x)}{(3x^2+1)^2}$.

   (d) Find the sign chart for $f''(x)$.
   Solution: $f''(x) < 0$ on $(-\infty, -\sqrt{1/3})$ and on $(\sqrt{1/3}, \infty)$ and positive on $(-\sqrt{1/3}, \sqrt{1/3})$, as shown on the sign chart for f'':



   (e) Find the intervals over which f is concave upwards.
   Solution: From (c) it follows that f is concave upwards on $(-\sqrt{1/3}, \sqrt{1/3})$.

10. (12 points) Compound Interest.

   (a) Consider the equation $2000(1 + 0.03)^{4t} = 6000$. Find the value of t and interpret your answer in the language of compound interest.
   Solution: t is the time required for an investment at rate $r = 12\%$ compounded quarterly to triple. Use logs to get $t = 9.29$ years.

   (b) Consider the equation $P(1 + 0.04)^{4\cdot 10} = 5000$. Solve for P and interpret your answer in the language of compound interest.
   Solution: P is the principle in dollars required to grow a 16% investment compounded quarterly over 10 years to grow to \$5000. Another way to say this is that P is the present value of \$5000 compounded quarterly over 10 years. Solve the equation to get $P = \$1041.45$

(c) Consider the equation $Pe^{10r} = 2P$. Solve for r and interpret your answer in the language of compound interest.

Solution: We're compounding continuously, and getting twice the original investment. If we interpret the r as rate, we're asking what rate of interest will cause a continuously compounded 10-year investment to double. Solve $10r = \ln 2$ to get $r = 0.069$ or 6.9%.