November 15, 2012 **Name** ______________________________

The total number of points available is 158. Throughout this test, **show your work.** Using a calculator to circumvent ideas discussed in class will generally result in no credit. In general you should carry out calculations to three decimal places.

1. (15 points) Consider the function $f(x) = (x^2 - 4x + 4)e^{2x}$.

   (a) Use the product rule to find $f'(x)$.
   Solution: $f'(x) = (2x - 4)e^{2x} + 2(x^2 - 4x + 4)e^{2x}$

   (b) List the critical points of f.
   Solution: Factor the expression above to get $(2x^2 - 8x + 8 + 2x - 4)e^{2x} = 2e^{2x}(x - 2)(x - 1)$, which has value 0 when $x = 2, x = 1$.

   (c) Construct the sign chart for $f'(x)$.
   Solution: f' is positive on $(-\infty, 1)$ and on $(2, \infty)$.

   (d) Write in interval notation the interval(s) over which f is increasing.
   Solution: f is increasing on $(-\infty, 1)$ and on $(2, \infty)$.

2. (15 points) Consider the function $f(x) = \ln[(2x^2+3)(7x-2)(x^2-4)]$.

   (a) Recalling that $\ln(x)$ is defined precisely when $x > 0$, find the domain of f.

   Solution: Build the sign chart for the function $g(x) = (2x^2+3)(7x-2)(x^2-4) = (2x^2+3)(7x-2)(x-2)(x+2)$ to see that g is positive on $(-2, 2/7)$ and $(2, \infty)$. So the domain of the function f is the union of these two sets.

   (b) Let $g(x) = (2x^2+3)(7x-2)^2(x^2-4)^2$. Use logarithmic differentiation to find g'. You need not simplify your answer.

   Solution: Take logs of both sides to get $\ln g(x) = \ln(2x^2+3)(7x-2)^2(x^2-4)^2$. This simplifies to $\ln g(x) = \ln(2x^2+3) + 2\ln(7x-2) + 2\ln(x^2-4)$. Now taking the derivative of both sides yields

$$\frac{g'(x)}{g(x)} = \frac{4x}{2x^2+3} + \frac{2 \cdot 7}{7x-2} + \frac{2 \cdot 2x}{x^2-4}$$

   Finally, we can write

$$g'(x) = (2x^2+3)(7x-2)^2(x^2-4)^2 \left(\frac{4x}{2x^2+3} + \frac{14}{7x-2} + \frac{4x}{x^2-4} \right).$$

   (c) Compute $g(1)$, $g'(1)$ and find an equation for the line tangent to g at the point $(1, g(1))$.

   Solution: First note that $g(1) = 5 \cdot 5^2 \cdot 9 = 1125$ and $g'(1) = g(1)[\frac{4}{5} + \frac{14}{5} - \frac{4}{3}] = 1125 \cdot 34/15 = 2550$.

3. (25 points) Consider the function $f(x) = \ln(x^2 + 4)$.

   (a) Find the domain of f.

   Solution: f is defined for all x because $x^2 + 4$ is positive for all x.

   (b) Find both $f'(x)$ and $f''(x)$.

   Solution: $f'(x) = \frac{2x}{x^2+4}$ and, by the quotient rule, $f''(x) = \frac{2(x^2+4)-2x(2x)}{(x^2+4)^2}$, which simplifies to

$$f''(x) = \frac{-2x^2 + 8}{(x^2 + 4)^2}.$$

   So $f''(x)$ has two zeros, $x = 2$ and $x = -2$.

   (c) Build the sign chart for $f''(x)$.

   Solution: The sign chart shows that $f''(x)$ is positive only between -2 and 2.

   (d) Does f have an absolute maximum or and absolute minimum? Is so, which? ... and where?

   Solution: f has an absolute minimum at $x = 0$. $f(0) = \ln 4$.

(e) Find an interval over which f is increasing. As usual, to get credit for this, you must show your work.

Solution: Since f' is positive on $(0, \infty)$, it follows that f is increasing on that interval.

(f) Does f have any inflection points?

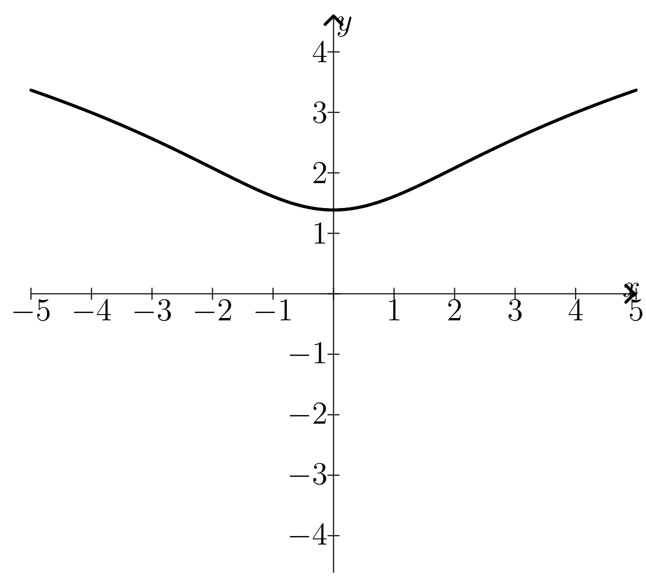
Solution: Yes, at both $(-2, \ln 8)$ and $(2, \ln 8)$.

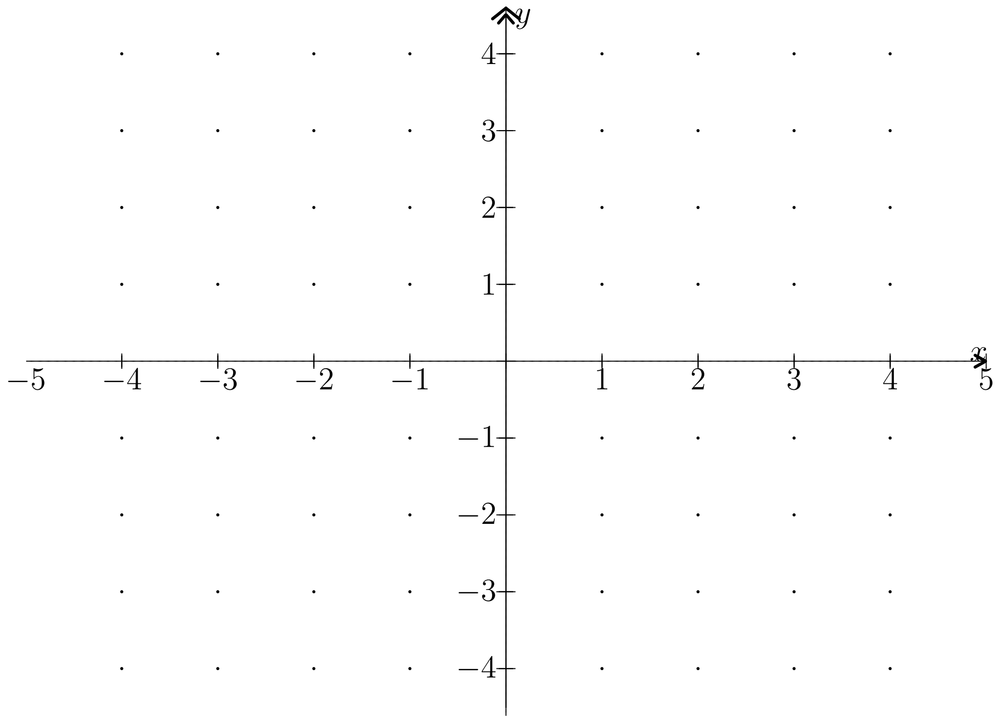
(g) Use the information from part b. to find the intervals where f is concave upwards.

Solution: Since f'' is positive on $(-2, 2)$, f is concave upwards on that interval.

(h) Sketch the graph of f.

Solution:

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4. (20 points) According to Newton's Law of Cooling, the rate at which an object's temperature changes is proportional to the difference between the object's temperature and that of the medium into which it is emersed. If $F(t)$ denotes the temperature of a cup of instant coffee (initially $212°F$), then it can be proven that

$$F(t) = T + Ae^{-kt},$$

where T is the air temperature, $68°F$, A and k are positive constants, and t is expressed in minutes.

   (a) What is the value of A?

   Solution: Since $F(t) = 68 + Ae^{-kt}$, it follows that $F(0) = 68 + A \cdot 1 = 212$ so $A = 144$.

   (b) Suppose that after exactly 8 minutes, the temperature of the coffee is $136.6°F$. What is the value of k?

   Solution: Solve $F(t) = 136.6 = 68 + 144e^{-k(8)}$ for k to get $k = 0.09269$.

   (c) Use the information in (a) and (b) to find the number of minutes before the coffee reaches the temperature of $80°F$.

   Solution: Solve the equation $80 = 68 + 144e^{-0.09269t}$ for t to get first $e^{-0.09269t} = 12/144 = 0.83333$, and taking logs of both sides yields $t = 26.8$ minutes.

5. (10 points) Find an equation for the line tangent to the graph of $y = \ln x^2$ at the point $(1, 0)$.

   Solution: The derivative of the function is $y' = 2/x$ so the slope of the line at $x = 1$ is $2/1 = 2$ and the line is $y - 0 = 2(x - 1)$. That is $y = 2x - 2$.

6. (10 points) Find an equation for the line tangent to the graph of $y = e^{5x-2}$ at the point $(2/5, 1)$.

   Solution: $y' = 5(e^{5x-2}$ so $m = 5e^{5\cdot\frac{2}{5}-2} = 5$. Thus the line is $y-1 = 5(x-2/5)$. and in slope-intercept form $y = 5x - 1$.

7. (12 points) A skull from an archeological dig has one-twelfth the amount of Carbon-14 it had when the specimen was alive.

   (a) Recall that the half-life of Carbon-14 is 5770 years. Find the decay constant k.

   Solution: We must solve the equation $Q(5770) = Q_0/2$ for k, where $Q(t) = Q_0e^{-kt}$. The equation leads to $e^{-5770k} = 1/2$ which means that $k \approx 1.201 \times 10^{-4} = 0.00012$.

   (b) What is the age of the specimen? Round off your answer to the nearest multiple of one hundred years.

   Solution: To solve the equation $Q_0e^{-kt} = Q_0/12$, divide both sides by Q_0 to get $e^{-kt} = 1/12$. This happens for $t = 20685$ years, which rounds to 20700.

8. (12 points) Compound Interest.

   (a) Consider the equation $3000(1 + 0.03)^{4t} = 6000$. Find the value of t and interpret your answer in the language of compound interest.

   Solution: t is the time required for an investment at rate $r = 12\%$ compounded quarterly to double. Use logs to get $t = 5.86$ years.

   (b) Consider the equation $P(1 + 0.04)^{4\cdot 10} = 5000$. Solve for P and interpret your answer in the language of compound interest.

   Solution: P is the principle in dollars required to grow a 16% investment compounded quarterly over 10 years to grow to \$5000. Another way to say this is that P is the present value of \$5000 compounded quarterly over 10 years. Solve the equation to get $P = \$1041.45$

   (c) Consider the equation $Pe^{10r} = 3P$. Solve for r and interpret your answer in the language of compound interest.

   Solution: We're compounding continuously, and getting triple the original investment. If we interpret the r as rate, we're asking what rate of interest will cause a continuously compounded 10-year investment to **triple**. Solve $10r = \ln 3$ to get $r = 0.1099$ or 11%.

9. (30 points) Spreading of a rumor. Three hundred college students attend a lecture of the Dean at which she hints that the college will become coed. The rumor spreads according to the logistic curve

$$Q(t) = \frac{3000}{1 + Be^{-kt}},$$

where t is measured in hours.

(a) Compute the parameter B.

Solution: Note that $Q(0) = 300 = \frac{3000}{1+Be^{-k0}}$, so $1 + B = 10$, and $B = 9$.

(b) How many students attend the college? Hint: the question is not 'how many attended the lecture?'

Solution: Take the limit as $t \to \infty$ since eventually everyone knows the rumor. $\lim_{t\to\infty} \frac{3000}{1+9e^{-kt}} = 3000$.

(c) Two hours after the speech, 600 students had heard the rumor. How many students had heard the rumor after four hours?

Solution: Solve $Q(2) = 600 = \frac{3000}{1+9e^{-2k}}$ for k. We get $9e^{-2k} = 4$ and $k = \frac{\ln 4 - \ln 9}{-2} \approx 0.40546$. Therefore $Q(4) = \frac{3000}{1+9e^{-4k}} = 1080$.

(d) How fast is the rumor spreading after four hours?

Solution: We need $Q'(t)$. So write $Q(t) = 3000(1 + 9e^{-kt})^{-1}$ and use the chain rule to get $Q'(t) = \frac{3000 \cdot 9ke^{-kt}}{(1+9e^{-kt})^2}$. Using a calculator, we find that $Q'(4) = \frac{27000ke^{-4k}}{(1+9e^{-4k})^2} \approx \frac{2162.48}{7.716} \approx 280$ students per hour.

(e) After how many hours will 2000 students have heard the rumor?

Solution: We need to solve the equation $Q(t) = 2000 = \frac{3000}{1+9e^{-kt}}$ for t. We have $1 + 9e^{-kt} = 3/2$, and it follows that $e^{-kt} = 1/18$. We can solve this for t to get $t = \frac{\ln 1 - \ln 18}{-k} = \frac{\ln 18}{k} \approx 7.128$ hours.