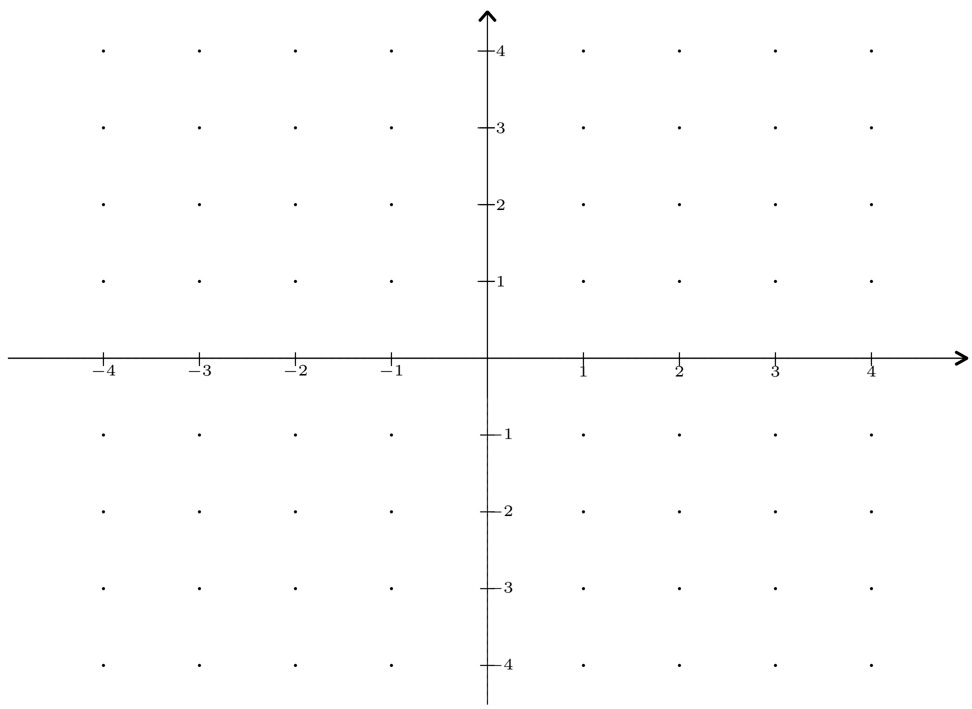
April 10, 2013 **Name** ______________________________

The total number of points available is 145. Throughout this test, **show your work.** Using a calculator to circumvent ideas discussed in class will generally result in no credit.

1. (25 points) Consider the rational function $r(x) = \frac{(x^2-4)(x+3)(x)}{(2x-4)(x^2-9)(x+1)}$. Find the vertical and horizontal asymptotes, and the zeros of the function. Build the sign chart for the function. Sketch the graph on the coordinate system provided. Use the graph to state the intervals over which the function is increasing.



Solution: The function reduces to $r(x) = \frac{(x+2)x}{2(x-3)(x+1)}$. From here we can read off the zeros: $x = -2$ and $x = 0$; the vertical asymptotes: $x = 3$ and $x = -1$, and the horizontal asymptote $y = 1/2$. The function is decreasing over each of the intervals $(-\infty, -1), (-1, 3)$ and $(3, \infty)$.

2. (10 points) Find the intervals over which $f(x) = 0.25x + x^{-1}$ is increasing.

   Solution: Solve the equation $f'(x) = 0.25 - 1/x^2 = 0$ to get two critical points $x = \pm 2$. Include the vertical asymptote $x = 0$, and then build the sign chart for f' to see that f is increasing on both $(-\infty, -2)$ and $(2, \infty)$.

3. (10 points) Find the interval(s) over which $g(x) = 2x^3 + 3x^2 - 12x + 7$ is concave upwards.

   Solution: Since $g'(x) = 6x^2 + 6x - 12$ and $g''(x) = 12x + 6$, the sign chart for g'' reveals that g is concave upwards on $(-1/2, \infty)$.

4. (20 points) Find all critical points of $H(x) = (2x+2)^3 \cdot x^5$. Then identify each critical point as the location of a local maximum, local minimum, or neither.

   Solution: By the product rule, $H'(x) = 3(2x+2)^2 \cdot 2x^5 + 5x^4(2x+2)^3 = (2x+2)^2 x^4[6x+5(2x+2)]$, so the zeros are $x = -1, x = 0$, and $x = -10/16 = -5/8$. But notice that there is no sign change at either -1 or 0. At $x = -5/8$, H' changes from negative to positive, so the point $x = -5/8$ is the (only) location of a local minimum.

5. (20 points) If 300 square inches of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

   Solution: It follows that $x^2 + 4xy = 300$. As a result, we need to maximize $V = x^2y = x^2(300 - x^2)/4x = (1/4)(300x - x^3)$. Thus $v'(x) = (300 - 3x^2)/4$ which is zero at $x = \pm 10$. Of course -10 is a nonsense answer. Thus the largest possible volume is $V(10) = (300 \cdot 10 - 10^3)/4 = 2000/4 = 500$ cubic inches.

6. (12 points) Find the time it takes for an 8% investment compounded quarterly to triple in value. Round off to the nearest tenth of a year.

   Solution: Since $A = P(1 + \frac{r}{n})^{nt}$, we have $3P = P(1 + \frac{0.08}{4})^{4t}$, and therefore $t = \frac{\ln(3)}{4\ln(1.02)} \approx 13.9$ years.

7. (12 points) Nuclear Fallout. Strontium 90 (Sr-90) is a radioactive isotope of strontium. It is present in the fallout after a nuclear explosion. It is especially hazardous to human and other animals because, upon ingestion of contaminated food, it is absorbed into the bone structure. Its half-life is 27 years.

   (a) If the concentration of Sr-90 is four times the 'safe' level, in how many years will the concentration be at the safe level?

   Solution: To get to the safe level, the concentration must be reduced by a factor of four, which takes 54 years.

   (b) If the concentration of Sr-90 is four times the safe level, how many years are needed to reduce the concentration to one third the safe level?

   Solution: The model for exponential decay is $Q(t) = Q_0e^{-kt}$. Since the half-life is 27 years, we have $Q(27) = Q_0/2 = Q_0e^{-27k}$, which we can solve for k to get $k = -\ln(2)/-27 \approx 0.02567$. Now, solving $Q(t) = Q_0/12$, we get $(1/2)^{t/27} = 1/12$, so $t \approx 96.794$ years.

8. (12 points) Find an equation for the line tangent to the graph of $g(x) = x^2 e^{2x}$ at the point $(1, g(1))$.

   Solution: First note that $g'(x) = 2xe^{2x} + x^2e^{2x} \cdot 2 = e^{2x}(2x + 2x^2)$. So $g'(1) = 4e^2$ and $g(1) = e^2$. The tangent line is $y - e^2 = 4e^2(x - 1)$.

9. (12 points) Find an interval over which the function $G(x) = \ln(x^2 + 1)$ is concave upwards.

   Solution: First, $G'(x) = \frac{2x}{x^2+1}$ and $G''(x) = \frac{2(x^2+1)-2x\cdot 2x}{(x^2+1)^2}$. The sign of G'' depends only on the numerator, which is positive precisely between -1 and 1.

10. (12 points) Find an equation for the line tangent to the graph of $h(x) = \sqrt{x^2+3} \cdot (2x-3)^4 \cdot (3x+1)^{-1}$ at the point $(1, h(1))$.

    Solution: First note that $h(1) = 2 \cdot 1 \cdot 1/4 = 1/2$. Use logarithmic differentiation to get $g'(x) = \frac{1}{2}\ln(x^2 + 3) + 4\ln(2x - 3) - \ln(3x + 1)$, which has value $-17/4$ at $x = 1$. So the line is given by $y - 1/2 = -17(x - 1)/4$.