November 21, 2013 **Name** ______________________________

The problems count as marked. The total number of points available is 153. Throughout this test, **show your work.** Using a calculator or a technique like L'hospital's Rule to circumvent ideas discussed in class will generally result in no credit.

A (15 points) Let $f(x) = \ln(2x^2 - 3x)$. Notice that the domain of f includes the point $x = 2$, since $8 - 6 = 2 > 0$. Please leave your answers in terms of ln, that is, do not use your calculator to convert to decimals.

(a) What is the domain of f? Express your answer in interval notation.
Solution: Solve $2x^2 - 3x = x(2x-3) > 0$ to get as the domain $(-\infty, 0) \cup (3/2, \infty)$.

(b) Find $f'(x)$.
Solution: $f'(x) = \frac{4x-3}{2x^2-3x}$.

(c) Find $f'(2)$.
Solution: $f'(2) = \frac{8-3}{8-6} = 5/2$.

(d) Find an equation for the line tangent to the graph of f at the point $(2, f(2))$.
Solution: $y - \ln(2) = 5(x-2)/2$.

1. (8 points) How long does it take an 8% investment, compounded continuously, to triple in value?

   Solution: Solve the equation $Pe^{rt} = 3P$ for t where P doesn't matter and $r = 0.08$. You get $e^{0.08t} = 3$ and finally $t = \ln(3)/0.08 \approx 13.71$ years.

2. (15 points) Let $P = (0, 0)$. For each line listed below, find the point on the line that is closest to P.

   (a) $x = 3$

   Solution: Since $x = 3$ is a vertical line, the point on $x = 3$ closest to $(0, 0)$ is $(3, 0)$.

   (b) $y = 5$

   Solution: Since $y = 5$ is a horizontal line, the point on $y = 5$ closest to $(0, 0)$ is $(0, 5)$.

   (c) $y = 2x - 7$

   Solution: As we discussed in class, you can minimize the square of the distance, $D(x) = x^2 + (2x - 7)^2$. Thus $D'(x) = 2x + 2(2x - 7) \cdot 2$, by the chain rule, and this linear function has one zero, $x = 2.8$. The y coordinate is $y = 2(2.8) - 7 = -1.4$, which is consistent with what we saw in class, the lines $y = 2x - 7$ and $y = -x/2$ are perpendicular since the product of their slopes is $2 \cdot (-1/2) = -1$.

3. (20 points) Function f has been differentiated and the result is the function $f'(x)$ below.

$$f'(x) = \begin{cases} x + 10 & \text{if } x < 0 \\ -x^2 + 4x - 3 & \text{if } 0 < x < 10 \\ x - 30 & \text{if } 10 < x \end{cases}$$

   (a) What is the domain of f'?

   Solution: $(-\infty, 0) \cup (0, 10) \cup (10, \infty)$.

   (b) Find the intervals over which the function f' is increasing.

   Solution: Note that f' is increasing on $(-\infty, 0)$ and $(0, 2)$ and $(10, \infty)$.

   (c) Find the intervals over which the function f is increasing.

   Solution: The function f is increasing precisely where f is positive, namely $(-10, 0), (1, 3)$, and $(30, \infty)$.

B (15 points) For a particular person learning to type, it was found that the number N of words per minute the person was able to type after t hours of practice, was given by

$$N = N(t) = 100(1 - e^{-0.02t}).$$

(a) After 10 hours of practice how many words per minute could the person type?

Solution: $N(10) = 100(1 - e^{-0.02(10)}) \approx 18.12.$

(b) After 11 hours of practice how many words per minute could the person type?

Solution: $N(11) = 100(1 - e^{-0.02(11)}) \approx 19.74.$

(c) What was the rate of improvement after 10 hours of practice? Please include the units in your answer. How does this number relate to the answers to parts a and b.

Solution: $N'(t) = 100(0 + .02e^{-0.02t} = 2e^{-0.02t})$ so $N'(10) \approx 1.637$ words per minute per hour of practice time.

(d) What was the rate of improvement after 40 hours of practice?

Solution: $N'(40) = 2e^{-0.8} \approx 0.898$ words per minute per hour.

4. (30 points) A manufacturer has been selling 1300 television sets a week at \$450 each. A market survey indicates that for each \$27 rebate offered to a buyer, the number of sets sold will increase by 270 per week. In other words, if they drop the price by \$27, they sell 270 more sets, etc. Notice that p is a linear function of x since the slope $m = -1/10$ is constant.

   (a) At what price would the company sell exactly 1030 sets per week? At what price would the company sell exactly 1570 sets per week?

   Solution: At 1030 sets, the price would have to be $450+27 = 477$, and to sell 1570 sets they would have to reduce the price by \$27 to $450-27 = 423$.

   (b) Calculate the revenue associated with sales of $x = 1300$ and $x = 1570$ TV sets sold.

   Solution: $1300 \cdot \$450 = \$585,000$ and $1570 \cdot \$423 = \$664110$.

   (c) Find the demand function $p(x)$, where x is the number of the television sets sold per week, and $p(x)$ is measured in dollars. In other words, find the relationship between price and number sold.

   Solution: $-\frac{1}{10}(x-1300)+450$. Note that $p(1300) = 450$ and $p(1570) = 423$, so the slope of the linear function $p(x)$ is $\frac{450-423}{1300-1570} = -\frac{1}{10}$. Therefore $p(x) = -\frac{1}{10}(x-1300)+450$.

   (d) How large rebate should the company offer to a buyer, in order to maximize its revenue?

   Solution: $x = 2900$. $R(x) = xp(x) = -\frac{x}{10}(x-1300)+450x$. Differentiate $R(x)$ to find the critical points, in this case just $x = 2900$.

   (e) Suppose the weekly cost function is $97500 + 150x$. Calculate the profit associated with sales of $x = 1300$ and $x = 1570$, TV sets during a week.

   Solution:

   (f) How should it set the size of the rebate to maximize its profit?

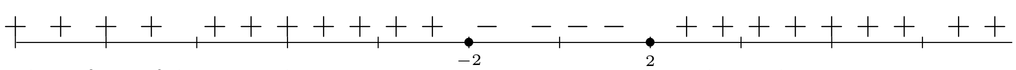
   Solution: \$85. Note that $P(x) = R(x) - C(x)$ and $P'(x) = R'(x) - C'(x) = -x/10 - (x-1300)/10 + 450 - 150$ which is zero when $x = 2150$, so this is the number of TV sets that maximizes the profit. This corresponds to a price per set of $p = 365$ which means the rebate must be $\$450 - \$365 = \$85$.

5. (20 points) Consider the function $f(x) = e^{4x^3-48x}$.

   (a) Compute $f'(x)$.
   Solution: $f'(x) = (12x^2 - 48)e^{4x^3-48x}$.

   (b) Find the critical points of f.
   Solution: $(12x^2 - 48) = 12(x-2)(x+2) = 0$ at $x = 2$ and $x = -2$.

   (c) Find the intervals over which f is increasing.
   Solution: The sign chart for f' is



   Therefore f has a relative maximum at $x = -2$ and a relative minimum at $x = 2$.

   (d) (5 points) Compute $f''(x)$ and discuss the concavity of f. You might have to use your graphing calculator for this part of the problem. Do not spend much time with this part. Its only worth 5 points. In case you don't have a graphing calculator, you can get points by invoking the Intermediate Value Theorem.

   Solution: The second derivative of f is given by the product rule: $f''(x) = e^{4x^3-48x}(24x + (12x^2 - 48)^2)$. Use the trace feature on the graphing calculator to find that there are two zeros of f'', roughly $x = -1.28$ and $x = -0.73$. f'' is negative between these two values, and positive otherwise. Thus, f is concave down on $(-1.28, -0.73)$ and concave up otherwise.

6. (30 points) Let $r(x)$ be the rational function defined by $r(x) = \frac{3x^2(x-3)}{x(x^2-4)}$.

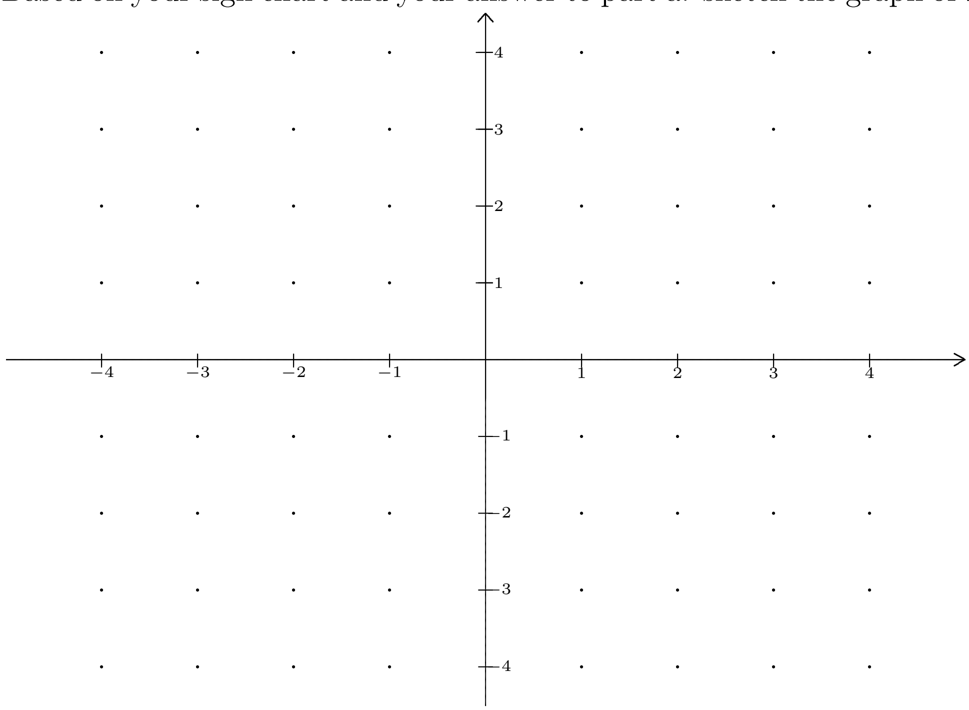
(a) List the zeros of the function and the vertical asymptotes, being clear to distinguish which is which. Also, discuss any horizontal asymptote, again making sure to distinguish it from the other items in the question.

Solution: There are two zeros, $x = 0$ and $x = 3$. There are two vertical asymptotes, $x = 2$ and $x = -2$. The horizontal asymptote is $y = 3$.

(b) Build the sign chart for $r(x)$.

Solution: The function r is positive on $(-\infty, -2)$, $(0, 2)$ and $(3, \infty)$.

(c) Based on your sign chart and your answer to part a. sketch the graph of r.



(d) Based on your graph in part c, build the sign chart for $r'(x)$. Explain how you got this. A calculator solution is not acceptable here.

Solution: The only reasonable way to draw the graph shows that our function is increasing on each of the intervals $(-\infty, -2)$, $(-2, 2)$ and on $(2, \infty)$.