Let $U = \{0,1,2,3,4,5,6,7,8,9\}$ be the universal set, $A = \{0,1,2,3,4,5\}$, and $B = \{4,5,6,7\}$. Find each of the following numbers.

1. $|A \times B|$

   Solution: $|A \times B| = |A| \times |B| = 6 \times 4 = 24$.

2. $|\overline{A} \times \overline{B}|$

   Solution: $|\overline{A} \times \overline{B}| = |\overline{A}| \times |\overline{B}| = 4 \times 6 = 24$.

3. $|A \times \overline{B}|$

   Solution: $|A \times \overline{B}| = |A| \times |\overline{B}| = 6 \times 6 = 36$.

4. $|\overline{A} \times B|$

   Solution: $|\overline{A} \times B| = |\overline{A}| \times |B| = 4 \times 4 = 16$.

5. $|A \times B| + |\overline{A} \times \overline{B}| + |A \times \overline{B}| + |\overline{A} \times B|$

   Solution: $|A \times B| + |\overline{A} \times \overline{B}| + |A \times \overline{B}| + |\overline{A} \times B| = 24 + 24 + 36 + 16 = 100$. Another way to see this is to note that $U \times U$, which has 100 elements, can be decomposed (partitioned) into the four sets $A \times B, \overline{A} \times \overline{B}, A \times \overline{B}$ and $\overline{A} \times B$

6. $|(A \cap B) \times (A \cup B)|$

   Solution: $|(A \cap B) \times (A \cup B)| = |(A \cap B)| \times |(A \cup B)| = 2 \times 8 = 16$.

7. $|(A \cap B) \times \overline{(A \cup B)}|$

   Solution: $|(A \cap B) \times \overline{(A \cup B)}| = 2 \times 2 = 4$.

8. $|\{T : T \subseteq U, |T \cap A| = 2 \text{ and } |T \cap B| = 2\}|$. In other words, how many subsets T of U have exactly two elements in common with A and exactly two elements in common with B. Hint: start this problem by listing a few of the sets with the desired properties, and then ask yourself 'how much freedom is there in choosing such sets?'

   Solution: We'll look at three types of sets based on how T intersects $A \setminus B, A \cap B$, and $B \setminus A$. Case 1. $|T \cap A \cap B| = 2$, There are just four subsets of this type, $\{4,5\}, \{4,5,8\}, \{4,5,9\}$, and $\{4,5,8,9\}$. Next suppose $|T \cap (A \setminus B| = |T \cap (B \setminus A| = |T \cap (A \cap B| = 1$ There are 4 ways to choose the member of $A \setminus B$, 2 ways to choose the member of $A \cap B$, 2 ways to choose the member of $B \setminus A$ and 4 ways to attach a subset of $\{8,9\}$. Finally we could have $|T \cap (A \setminus B)| = 2 = |T \cap (B \setminus A)|$, in which case there are 6 ways to choose the two elements of A and 1 way to choose the 2 elements of B, and then 4 ways to finish building the set. The sum of these is $4 + 64 + 24 = 92$ ways to build T.