1. Find the least common multiple and the greatest common divisor of each of the following pairs of integers.

    (a) $m = 2001$ and $n = 1001$.

    Solution: The two numbers have no prime factors in common, so their GCD is 1 and their LCM is their product, 2003001.

    (b) $m = 2^3 \cdot 3^2 \cdot 5$ and $n = 3^2 \cdot 7^2$.

    Solution: LCM $= 2^3 3^2 5 \cdot 7^2$ and the GCD$= 3^2$.

    (c) $m = p^3 \cdot q^2 \cdot r$ and $n = p^2 \cdot q \cdot r^3$, where p, q, and r are different primes.

    Solution: $gcd = p^2 qr$ and $lcm = p^3 q^2 r^3$.

2. Notice that we can find the number of positive integer divisors of an integer N in terms of its prime factorization. For example, the number $2^i 3^j 5^k$ is a divisor of $2^l 3^m 5^n$ if and only if $i \le l, j \le m$ and $k \le n$. Find the number of divisors of each of the integers.

    (a) $m = 2001$.

    Solution: $2001 = 3 \cdot 23 \cdot 29$, so it has $(1+1)(1+1)(1+1) = 8$ divisors.

    (b) $n = 1001$.

    Solution: This one too factors into a product of three distinct primes, and therefore has 8 divisors.

    (c) $m = 2^3 \cdot 3^2 \cdot 5$.

    Solution: The number of divisors is $D = (3+1)(2+1)(1+1) = 24$.

    (d) $n = 3^2 \cdot 7^2$.

    Solution: The number of divisors is $D = (2+1)(2+1) = 9$.

    (e) $m = p^3 \cdot q^2 \cdot r$, where p, q, and r are different primes.

    Solution: The number of divisors is $D = (3+1)(2+1)(1+1) = 24$.

3. A high school with 1000 lockers and 1000 students tries the following experiment. All lockers are initially closed. Then student number 1 opens all the lockers. Then student number 2 closes the even numbered lockers. Then student number 3 changes the status of all the lockers numbered with multiples of 3. This continues with each student changing the status of all the lockers which are numbered by multiples of his or her number. Which lockers are closed after all the 1000 students have done their jobs?

    Solution: Build a table for the first 20 lockers, and notice that the lockers that end up open are those numbered $1, 4, 9$, and 16. This looks like the

squares. Think about what it takes to make a locker end up open. It takes and odd number of changes, which means an odd number of divisors. We know how to count the number of divisors of a number $N = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$. N has $D_N = (e_1 + 1)(e_2 + 1) \cdots (e_n + 1)$ divisors. So the issue is how can this last number be odd. Its odd if each factor $e_i + 1$ is odd, which means all the e_i have to be even. This is true precisely when N is a perfect square.

4. Using the Euclidean Algorithm, evaluate $g = GCD(4144, 7596)$ and $h = LCM(4144, 7596)$. Then solve the equation $g = 4144x + 7596y$, for integers x and y.

   Solution: Notice that the gcd of the two numbers is 4. We can either proceed as usual or we can divide both numbers by 4 and then solve the resulting problem. Lets try solving $1899x + 1036y = 1$ using the EA. Write down all the divisions and begin the unwinding operations.

$$
\begin{aligned}
1 &= 171 - 85 \cdot 2 \\
&= 171 - 85 \cdot (173 - 171) \\
&= -85 \cdot 173 + 86 \cdot 171 \\
&= -85 \cdot 173 + 86 \cdot (863 - 4 \cdot 173) \\
&= 86 \cdot 863 - 344 \cdot 173 - 85 \cdot 173 \\
&= 86 \cdot 863 - 429 \cdot 173 \\
&= 86 \cdot 863 - 429 \cdot (1036 - 863) \\
&= 515 \cdot 863 - 429 \cdot 1036 \\
&= 515 \cdot (1899 - 1036) - 429 \cdot 1036 \\
&= 515 \cdot 1899 - 944 \cdot 1036
\end{aligned}
$$

   Multiplying both sides by 4 gives

$$4 = 515 \cdot 7596 - 944 \cdot 4144,$$

   so $x = -944$ and $y = 515$.

5. Let a, b, c, d, and e be digits satisfying $4 \cdot \underline{abcde4} = \underline{4abcde}$. Find all five of the digits.

   Solution: Let $x = \underline{abcde}$. Then $4(10x+4) = 400000+x$, from which it follows that $39x = 399984$ and $x = 10256$.

6. A 10×10 square is decomposed into exactly 75 squares of various (integer) sizes. How many 3×3 squares are in this decomposition?

Solution: The largest square cannot be as large as 5×5 because in this case, the total number of squares would be less than 75. Let $x, y, z,$ and w denote the number of squares of area $1, 4, 9,$ and 16 respectively. Then $x + 4y + 9z + 16w = 100$ and $x + y + z + w = 75$. Subtracting the later from the former yields $3y + 8z + 15w = 25$. Since there are no integer solutions to $3y + 8z = 10$, we may conclude that $w = 0$. There is only one solution to $3y + 8z = 25$, namely, $y = 3$ and $x = 2$.

7. A two-digit integer N that is not a multiple of 10 is k times the sum of its digits. The number formed by interchanging the the digits is m times the sum of the digits. What is the relationship between m and k?

   Solution: Note that $10a + b = k(a + b)$ and $10b + a = m(b + a)$. Adding the two equations together, it follows that $k + m = 11$.

8. The fido challenge: http://digicc.com/fido/

   Solution: Let $\underline{abcd}$ denote your four-digit number and let $\underline{uvwz}$ denote the rearranged number. Then $\underline{abcd} - \underline{uvwz} \equiv (a + b + c + d) - (u + v + w + z) = 0$ (mod 9), because the sets $\{a, b, c, d\}$ and $\{u, v, w, z\}$ are identical. Therefore any three digits of the difference $\underline{abcd} - \underline{uvwz}$ determines the fourth one, unless the fourth digit is 0 or 9, in which case, we do not know which it is.

9. The crystal cabobble challenge: http://www.sinotrading.us/crystalball.htm

   Solution: Notice that all the multiples of 9 except 90 and 99 have the same symbol. Why is $10a + b - (a + b)$ always a multiple of 9?

10. How many two-digit positive integers N have the property that the sum of N and the number obtained by reversing the order of the digits of N is a perfect square?

    Solution: Let $N = 10x + y$. Then $10x + y + 10y + x = 11(x + y)$ must be a perfect square. Since $1 \leq x + y \leq 18$, it follows that $x + y = 11$. There are eight such numbers: 29, 38, 47, 56, 65, 74, 83, and 92.

11. A check is written for x dollars and y cents, both x and y two-digit numbers. In error it is cashed for y dollars and x cents, the incorrect amount exceeding the correct amount by \$17.82. Find a possible value for x and y.

    Solution: Note that $100y + x - (100x + y) = 99(y - x) = 1782$. It follows that $y - x = 18$, so any pair of two-digit numbers that differ by 18 will work.

12. Solve the alpha-numeric problem $\underline{2abc} \times 4 = \underline{cba2}$, where a, b and c are decimal digits.

Solution: We have $4(2000+100a+10b+c) = 1000c+100b+10a+2$. It follows quickly that $c = 8$. Then $400a + (4b + 3)10 = 100b + 10a$. Thus $39a + 3 = 6b$. Only $a = 1$ works and it follows that $b = 7$.

13. The rightmost digit of a six-digit number N is moved to the left end. The new number obtained is five times N. What is N? Problems 14 though 16 also refer to 6-digit numbers.

    Solution: The number is 142857. Let $N = \underline{abcdef}$ and let $x = \underline{abcde}$. Then we have $5N = 5(\underline{abcdef}) = \underline{fabcde}$. Note that we have adopted the convention of underlining the digits of an integer in decimal notation. For example $\underline{ab} = 10a + b$. Note that $5(\underline{abcdef}) = 5(\underline{abcde0} + f) = 5(10x + f) = \underline{fabcde} = 100000f+x$. This leads to $50x-x = 100000f-5f$ or $49x = 99995f$. Factoring both sides gives $7^2x = 5 \cdot 7 \cdot 2857 \cdot f$, which is equivalent to $7x = 5 \cdot 2857 \cdot f$, from which it follows that $f = 7$. Therefore $x = 14285$ and $N = 142857$.

14. Repeat the same problem with the 5 changed to a 4. That is $4(\underline{abcdef}) = \underline{fabcde}$

    Solution: There are several solutions, $N = 153846, N = 102564, N = 205128$ and $N = 230769$.

15. Next, consider another transformation. This time move the leftmost digit to the right end and change the 5 to a 3.

    Solution: $N = 142857$ or $N = 285714$.

16. Consider the problem you get when you move the right TWO digits to the left end. Can this new number be exactly half the original number? Find all solutions and prove that you have them all.

    Solution: Let $N = \underline{abcdef}$ and let $x = \underline{abcd}$. Then $2(\underline{efabcd}) = \underline{abcdef}$ translates into $2(10000(10e+f)+x) = 100x+10e+f$. Solving this for x yields $x = \frac{19999(10e+f)}{98} = \frac{(19992+7)(10e+f)}{98} = \frac{204 \cdot 98+7}{98}(10e + f) = (204 + \frac{7}{98})(10e + f)$. Since x is an integer, the last equation requires that $10e + f$ be a multiple of 14. Trying $14, 28,$ and 42, we see that they all work. But $56, 70,$ and 84 all fail. Thus the values of N are $285714, 571428, 857142$.