1. Ashley noticed that the set of ages of her relatives, all of whom were whole numbers in the range 1 up to 100 inclusive, has the unusual property that no two of them multiplied together is a perfect square. What is the largest number of relatives Ashley could have?

   Solution: A relation R on a set S is a set of ordered pairs (x, y), where both x and y belong to S. If R is a relation on a set S, instead of writing $(x, y) \in R$, we write xRy, and say 'x is related to y.' Define a relation $\sim$ on the set $S = \{1, 2, 3, 4, \ldots, 100\}$ as follows:

   $$\text{For any } x, y \in S,\ x \sim y \text{ if } x \cdot y \text{ is a perfect square.}$$

   Thus, for example $1 \sim 1, 1 \sim 4, 2 \sim 8$, and $3 \sim 12$. Next note that $\sim$ satisfies

   (a) $x \sim x$ for all $x \in S$ (*reflexive* property),

   (b) $x \sim y$ implies $y \sim x$ for all $x, y \in S$ (*symmetry* property), and

   (c) $x \sim y$ and $y \sim z$ implies $x \sim z$ for all $x, y, z \in S$ (*transitive* property).

   Relations that satisfy all three properties above are called *equivalence relations*. If R is any relation on a set S and $x \in S$, define the *cell* of x, denoted $[x]$ as follows:

   $$[x] = \{y \mid xRy\}.$$

   In the case when R is an equivalence relation, the cells of the members of S have a special property: two cells, $[x]$ and $[y]$ are either identical or they are disjoint. In other words, $[x] = [y]$ or $[x] \cap [y] = \emptyset$. We leave the proof of this fact to the reader. Our relation $\sim$ is an equivalence relation. To see this, note that (a) $x \sim x$ for all x because $x \cdot x$ is a perfect square; (b) if $x \sim y$, then $y \sim x$ because $x \cdot y = y \cdot x$; and (c) if $x \sim y$ and $y \sim z$, then $x \cdot y$ and $y \cdot z$ are both perfect squares. It follows that $x \cdot z = xy^2z \div y^2$ is also a perfect square, so $x \sim z$.

   Finally, to answer the question, note that Ashley can have at most one relative whose age belongs to a given cell. The maximum number of relatives Ashley could have is the number of cells. To count the cells, we can list the members of $[1] = \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}$. Then find the smallest member of S not in one of the cells listed, and find its cell. Thus $[2] = \{2, 8, 18, 32, 50, 72, 98\}$. Continuing in this way, we find that there are 61 cells. Alternatively, note that the smallest member of each cell is square free. Here we count 1 as square free. Furthermore, each cell can contain at most one square free number because the product of distinct square free numbers cannot be a square. We can count the square free numbers less than or equal to 100 using the Inclusion/Exclusion Principle:

$$100 - \left\lfloor \frac{100}{2^2} \right\rfloor - \left\lfloor \frac{100}{3^2} \right\rfloor - \left\lfloor \frac{100}{5^2} \right\rfloor - \left\lfloor \frac{100}{7^2} \right\rfloor + \left\lfloor \frac{100}{2^2 3^2} \right\rfloor + \left\lfloor \frac{100}{2^2 5^2} \right\rfloor = 61$$

2. Some properties of relations. We describe below some important properties that relations might or might not have. A relation R on a set A is called

   R. Reflexive if $\forall x \in A, xRx$.

   S. Symmetric if $\forall x, y \in A, xRy \Rightarrow yRx$.

   A. Antisymmetric if $\forall x, y \in A, xRy$ and $yRx \Rightarrow x = y$.

   T. Transitive if $\forall x, y, z \in A, xRy$ and $yRz \Rightarrow xRz$.

   Now let $A = \{1, 2, 3\}$. There are $2^4 = 16$ subsets of $\{R, S, A, T\}$. Find a relation on A for each of these subsets. For example, consider the subset $\{S, A, T\}$. We seek to find a relation on $\{1, 2, 3\}$ that is symmetric, anti-symmetric, and transitive, and *not* reflexive. To keep the relation from being reflexive, we must exclude one of the three ordered pairs $(1, 1), (2, 2), (3, 3)$. However, two of these could be included. So lets try $H = \{(1, 1), (2, 2)\}$. Is this symmetric? Is it transitive? Is it antisymmetric? Sketch the digraph of the relation and notice that it has just two loops. After some thought, you'll decide that H is symmetric, antisymmetric, and transitive. There are 15 other subsets of $\{R, S, A, T\}$. Find a relation for each of these, or prove that certain combinations do not exist.

   Solution: Not all combinations are achievable. In all the examples that follow, the notation xy means (x, y).

   (a) $\overline{RSAT}$:$\{12, 23, 32\}$

   (b) $R\overline{SAT}$:$\{11, 12, 22, 23, 32, 33\}$

   (c) $\overline{R}S\overline{AT}$:$\{12, 21, 23, 32\}$

   (d) $\overline{RS}A\overline{T}$:$\{12, 23\}$

   (e) $\overline{RSA}T$:$\{12, 13, 23, 32, 22, 33\}$

   (f) $RS\overline{AT}$:$\{11, 12, 21, 22, 23, 32, 33\}$

   (g) $R\overline{S}A\overline{T}$:$\{11, 12, 22, 23, 33\}$

   (h) $\overline{RS}AT$:$\{12, 13, 23\}$

   (i) $R\overline{SA}T$:$\{11, 12, 13, 22, 23, 32, 33\}$

(j) $\overline{R}SA\overline{T}$: can't happen, see below.

(k) $\overline{R}S\overline{A}T$:$\{11, 12, 21, 22\}$

(l) $RSA\overline{T}$: can't happen, see below.

(m) $RS\overline{A}T$:$\{11, 12, 21, 22, 33\}$

(n) $R\overline{S}AT$:$\{11, 12, 13, 22, 23, 33\}$

(o) $\overline{R}SAT$:ϕ

Now consider a relation R that is symmetric and not transitive. To make R not transitive, we must have some ordered pair (x, y) in R with $x \neq y$. If R is symmetric, the (y, x) also belongs to R. But this implies that R is not antisymmetric. Thus, the combinations $RSA\overline{T}$ and $\overline{R}SA\overline{T}$ are impossible.