Throughout we use both the notations $\binom{n}{r}$ and C_r^n for the number $\frac{n!}{(n-r)!r!}$.

1. How many paths consisting of a sequence of horizontal and/or vertical line segments with each segment connecting a pair of adjacent letters in the diagram below, is the word **CONTEST** spelled out as the path is traversed from beginning to end?

						C						
					C	**O**	**C**					
				C	**O**	**N**	**O**	**C**				
			C	**O**	**N**	**T**	**N**	**O**	**C**			
		C	**O**	**N**	**T**	**E**	**T**	**N**	**O**	**C**		
	C	**O**	**N**	**T**	**E**	**S**	**E**	**T**	**N**	**O**	**C**	
C	**O**	**N**	**T**	**E**	**S**	**T**	**S**	**E**	**T**	**N**	**O**	**C**

Solution: Imagine spelling the word backwards starting with the **T** in the middle and working back to the **C**'s on the outside. Count the number of paths from T to each letter in the array to get the array:

						1						
					6	**1**	**6**					
				15	**6**	**1**	**6**	**15**				
			20	**10**	**5**	**1**	**5**	**10**	**20**			
		15	**10**	**6**	**4**	**1**	**3**	**6**	**10**	**15**		
	6	**5**	**4**	**3**	**2**	**1**	**2**	**3**	**4**	**5**	**6**	
1	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**

The sum of the numbers in the C positions is $64 + 64 - 1 = 127$.

2. Recall that a Yahtzee Roll is a roll of five indistinguishable dice.
   a. How many different Yahtzee Rolls are possible?
   Solution: The answer is $Y_5^6 = \binom{6+5-1}{5} = 252$.
   b. How many Yahtzee Rolls have exactly 3 different numbers showing?
   Solution: These rolls come in two types, *aaabc* and *aabbc*. There are 60 of each type for a total of 120. There are 6 rolls which have just one value, 60 that have two values, 60 that have four values and 6 that have five different values. Note that $6 + 60 + 120 + 60 + 6 = 252$.

3. How many numbers can be expressed as a sum of three distinct members of the set $\{4, 5, 6, 7, 8, 9, 10, 11, 12\}$?

   Solution: The smallest such number is $4 + 5 + 6 = 15$ and the largest is $10 + 11 + 12 = 33$, and there are 19 numbers in the list $15, 16, 17, \ldots, 33$.

4. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
   a. How many five element subsets does the set have?
   b. How many subsets of S have an odd number of members?
   c. How many subsets of S have 1 as a member?
   d. How many subsets have 1 as a member and do not have 2 as a member?

   Solution: a. There are $\binom{10}{5} = 252$ five element subsets. b. Exactly half the $2^{10} = 1024$ subsets have an odd number of elements, so the answer is 512. c. Again 512 or half the subsets. d. Here the answer is 256, which is one-fourth the number of subsets.

5. Imagine that the 4×7 grid of squares below represents the streets of a part of the city where you live. You must walk 11 blocks to get from the lower left corner at A to the upper right corner at B.

   (a) How many different 11 block walks are there?

   Solution: Each path can be coded as an 11 letter string, each letter of which is an u(for up) or an r(for right). There are $\binom{11}{4} = 330$ such strings.

   (b) How many 11 block walks avoid the terrible corner marked with the bullet?

   Solution: Notice that there are $\binom{6}{2} \cdot \binom{5}{2} = 15 \cdot 10 = 150$ ways to GO THROUGH the terrible corner, so the must be $330 - 150 = 180$ ways to avoid it.

   (c) How many 11 block walks go through the terrible corner?

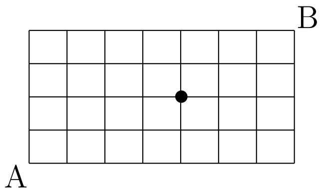
   Solution: See the previous problem.

   (d) How many different 12 block walks are there from A to B?

   Solution: There are none because each even unit walk must end on a vertex both of whose coordinates are even or both odd.

(e) How many different 13 block walks are there from A to B?

Solution: To solve this hard problem, note that each path of length 13 from A to B can be coded as a string of 13 letters. There are two types, one with 5u's, 1d, and 7r's, and the other with 4u's, 1l, and 8r's. But the d in the first type, which represents the one down move can appear only after the first u and before the last u. So pick the 6 positions for the u's and d and then pick one of the four middle positions for the d. There are $\binom{13}{6} \cdot 4 = 6864$ and of the second type there are $\binom{13}{9} \cdot 7 = 715 \cdot 7 = 5005$, for a total of 11869.



6. How many four-digit numbers have the property that the sum of the first three digits is the fourth digit. For example 1247 has the property.

   Solution: The solution below is incorrect because it fails to take into account the fact that four digit numbers must have a nonzero thousands digit. Let the number be $xyz(x+y+z)$, where x, y and z are digits and their sum is at most 9. For example when $x = y = 2$ and $z = 3$, we get the number 2237. If we could solve the inequality $x+y+z \leq 9$ or $x+y+z+w = 9$, where x, y, z, and w are all at least 0, we'd be done. This is a problem we can solve with the hot dog model. Take nine vertical bars and 3 dividers to code such a solution. For example, $||\diamond\diamond||||\diamond|||$ corresponds to the solution $x = 2, y = 0, z = 4$ and then to the number 2046. Therefore, there are $\binom{12}{3} = 220$ such numbers. It is not hard to modify this solution to solve the problem. We simply set aside the leftmost tally mark, not allowing a $\diamond$ to appear before it. This reduces the number of positions where the $\diamond$ can appear, so that there are $\binom{11}{3} = 165$ codes. Seeing a few examples will help. $|||\diamond|\diamond\diamond|||| \leftrightarrow 3104$ and $|\diamond||||\diamond||||\diamond \leftrightarrow 1449$.

7. How many numbers in the set $\{100, 101, 102, \ldots, 999\}$ have a sum of digits equal to 9? B. How many four digit numbers have a sum of digits 9? C. How many integers less than one million have a sum of digits equal to 9?

   Solution: Insert two dividers into a string of 8 counters to code such a three digit number. For example $||\diamond|||\diamond|||$ is a coding for 333 and $\diamond\diamond||||||||$ codes 108. There are $\binom{10}{2} = 45$ ways to code such a number. For four digit numbers, we get $\binom{10}{3} = 210$. C. These number are all at most 6 digits, so insert 5 dividers in a string of 9 vertical bars. For example $\diamond\diamond|||\diamond||\diamond\diamond||||$ represents the number 3204. There are $\binom{14}{9} = 2002$ ways to pick the 9 positions from the 14 locations.

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