March 26, 2004 **Name** ____________________________

There are 135 points available on this test. Each question is marked with its value. To get full credit for a problem, you must **show your work**. Correct answers with incorrect supporting work will receive substantially reduced credit.

1. (15 points) Let $p(x) = x^2 - 4x + 5$.

   (a) Compute $p'(x)$
   Solution: $p'(x) = 2x - 4$.

   (b) Compute $p''(x)$
   Solution: $p''(x) = 2$

   (c) Use the information in (a) to find an equation for the line tangent to the graph of p at the point $(1, 2)$.
   Solution: $y - 2 = p'(1)(x - 1) = -2(x - 1)$, so $y = -2x + 4$.

2. (20 points) Consider the *astroid* $x^{2/3} + y^{2/3} = 4$.

   (a) Show that the point $(-3\sqrt{3}, 1)$ belongs to the graph.
   Solution: $(-3\sqrt{3})^{2/3} + 1^{2/3} = (9 \cdot 3)^{1/3} + 1 = 4$.

   (b) Find y' as a function of x and y using implicit differentiation.
   Solution: Differentiate both sides to get $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{1/3} \cdot y' = 0$, so $y' = \frac{-\frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3}} = -(\frac{y}{x})^{1/3}$.

   (c) Find the slope of the line tangent to the curve at the point $(-3\sqrt{3}, 1)$.
   Solution: $m = -\left(\frac{1}{-3\sqrt{3}}\right)^{1/3} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \approx 0.5773$.

   (d) Find an equation for the tangent line whose slope you found above.
   Solution: Use the point-slope form to get $y - 1 = \frac{\sqrt{3}}{3}(x + 3\sqrt{3}) = \frac{\sqrt{3}x}{3} + 3$. Thus, $y = \frac{\sqrt{3}x}{3} + 4$

3. (30 points) Suppose the functions f and g are given partially by the table of values shown. The next problems refer to the functions f and g given in the tables. Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	3	2	5	2
1	3	5	2	6
2	5	3	4	1
3	6	4	3	4
4	4	6	1	5
5	1	3	2	4
6	1	2	5	3

(a) Let $K(x) = f \circ g(x)$. Compute $K'(3)$
Solution: $K'(3) = f'(g(3)) \cdot g'(3) = f'(3) \cdot g'(3) = 16$.

(b) Let $L(x) = f(x) \cdot g(x)$. Compute $L'(2)$.
Solution: $L'(2) = f'(2)g(2) + g'(2)f(2) = 17$.

(c) Let $U(x) = f \circ f(x)$. Compute $U'(1)$.
Solution: $U'(1) = f'(f(1)) \cdot f'(1) = f'(3) \cdot f'(1) = 4 \cdot 5 = 20$.

(d) Let $V(x) = g(x)/f(x)$. Compute $V'(4)$.
Solution: $V'(4) = \frac{g'(4)f(4) - f'(4)g(4)}{f(4)^2} = \frac{5 \cdot 4 - 6 \cdot 1}{16} = \frac{14}{16} = \frac{7}{8}$.

(e) Let $W(x) = (g(x))^2$. Compute $W'(5)$.
Solution: $W'(5) = 2g(5)g'(5) = 2 \cdot 2 \cdot 4 = 16$.

(f) Let $Z(x) = g(x^2 \cdot f(x))$. Compute $Z'(1)$.
Solution: $Z'(1) = g'(1^2 f(1))(2 \cdot 1 f(1) + f'(1) \cdot 1^2) = g'(3)(2 \cdot 3 + 5) = 4 \cdot 11 = 44$.

4. (25 points)

   (a) Find $\frac{d}{dx}(\sin x)$
   Solution: $\frac{d}{dx}(\sin x) = \cos x$

   (b) Write an equation involving the functions sin and $\sin^{-1}$, the composition operation, and the identity function. In other words write an equation that shows you know what $\sin^{-1} x$ is.
   Solution: $\sin \circ \sin^{-1}(x) = x$ or $\sin(\sin^{-1}(x)) = x$.

   (c) Differentiate both sides of the equation in (b).
   Solution: Let $y = \sin^{-1}(x)$. By the chain rule, $\cos(y) \cdot y' = 1$, so $y' = 1/\cos y = 1/\cos(\sin^{-1}(x))$.

   (d) Use the result in (c) to find an expression for $\frac{d}{dx}(\sin^{-1} x)$.
   Solution: Using the triangle with sides x, $\sqrt{1-x^2}$, and 1, it follows that $\frac{d}{dx}(\sin^{-1} x) = 1/\sqrt{1-x^2}$.

   (e) Let $h(x) = \sin^{-1}(x^2)$. Compute $h'(x)$.
   Solution: Using the chain rule, $h'(x) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \frac{2x}{\sqrt{1-x^4}}$.

5. (25 points) Compute the following derivatives.

   (a) $\frac{d}{dx}e^{\sin x}$
   Solution: $\frac{d}{dx}e^{\sin x} = e^{\sin x} \cdot \cos x$

   (b) $\frac{d}{dx}\ln(\tan x)$
   Solution: $\frac{d}{dx}\ln(\tan x) = \frac{1}{\tan x} \cdot \sec^2 x = \csc x \cdot \sec x$.

   (c) $\frac{d}{dx}\sqrt{x}(\ln x)$
   Solution: $\frac{d}{dx}\sqrt{x}(\ln x) = \frac{1}{2}x^{-1/2}\ln x + \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}\left(\frac{\ln x}{2} + 1\right)$.

   (d) $\frac{d}{dx}(\cos(x^2))^3$
   Solution: This is a triple composition, so you use the chair rule twice: $\frac{d}{dx}(\cos(x^2))^3 = -3(\cos x^2)^2 \cdot \sin x^2 \cdot 2x$.

   (e) $\frac{d}{dx}\tan^{-1}(2x)$
   Solution: $\frac{d}{dx}\tan^{-1}(2x) = \frac{1}{1+(2x)^2} \cdot 2 = \frac{2}{1+4x^2}$.

6. (20 points) Suppose f is defined by:

$$f(x) = \begin{cases} \ln(3x) & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

(a) Find $f'(3)$.
Solution: Near $x = 3$, $f'(x) = \frac{1}{3x} \cdot 3$, so $f'(3) = 1/3$.

(b) Find $f'(-e)$.
Solution: Near $-e$, $f'(x) = (1/-x) \cdot -1 = 1/x$, so $f'(-e) = -1/e$.

(c) Find an equation for the line tangent to the graph of f at the point $(-e, f(-e))$.
Solution: Since $m = -1/e$ and $f(-e) = \ln(-(-e)) = 1$, it follows that an equation for the line is $y - 1 = -\frac{x}{e} - 1$, or $y = -\frac{x}{e}$.