April 23, 2004 **Name** ____________________________

There are 115 points available on this test. Each question is marked with its value. To get full credit for a problem, you must **show your work**. Correct answers with incorrect supporting work will receive substantially reduced credit.

1. (20 points) A car A is traveling west at 40 miles per hour while car B is traveling north at 50 miles per hour. At exactly noon, car A is 3 miles east of an intersection P and car B is 4 miles south of P. At what speed are the cars moving toward each other?

   Solution: Orient the problem so that P is the origin. Then car A's position at noon is $(3, 0)$ and car B's position is $(0, -4)$. Since A is moving west, it follows that $\frac{dA}{dt} = -40$ mile per hour and $\frac{dB}{dt} = 50$ mile per hour. The distance between the cars is given by $D(t) = (A^2 + B^2)^{1/2}$. Thus

$$\begin{aligned}\frac{dD}{dt} &= \frac{1}{2}(A^2+B^2)^{-1/2}\cdot(2A\frac{dA}{dt}+2B\frac{dB}{dt})\\ &= \frac{1}{2}(3^2+(-4)^2)^{-1/2}\cdot(2\cdot 3(-40)+2\cdot(-4)\cdot 30)\\ &= \frac{1}{2}\frac{1}{5}(-240-400)\\ &= -640/10 = -64\end{aligned}$$

   miles per hour.

2. (20 points) A particle is moving along the curve $y^2 = x^3 - 2xy + 3x^2 + 1$.

   (a) Show that the point $(2, 3)$ belongs to the curve.
   Solution: Note that $3^2 = 2^3 - 2\cdot 2\cdot 3 + 3\cdot 2^2 + 1 = 9$.

   (b) Find the slope of the line tangent to the curve at $(2, 3)$.
   Solution: Differentiate both sides with respect to x to get $2yy' = 3x^2 - 2y - 2xy' + 6x$ and solving this for y' yields $y' = \frac{3x^2-2y+6x}{2y+2x}$ which we evaluate at $(2, 3)$ to get $y' = 3\cdot 2^2 - 2\cdot 3 + 6\cdot 2 \div (2\cdot 3 + 2\cdot 2) = 9/5$.

   (c) If $\frac{dx}{dt} = 5$ at the point $(2, 3)$, what is $\frac{dy}{dt}$ at $(2, 3)$?
   Solution: Differentiate both sides with respect to t to get $2yy' = 3x^2x' - 2x'y - 2xy' + 6xx'$. Replace $x' = \frac{dx}{dt}$ with 2 and replace x and y with their values to get $2\cdot 3\cdot y' = 3\cdot 2^2\cdot 5 - 2\cdot 5\cdot 3 - 2\cdot 2\cdot y' + 6\cdot 2\cdot 5$. Thus $6y' + 4y' = 60 - 30 + 60 = 90$ from which it follows that $\frac{dy}{dt} = 90/10 = 9$.

3. (20 points) Consider the function $f(x) = \frac{\cos x}{2+\sin x}$ defined over the interval $[0, 2\pi]$.

   (a) Find $f'(x)$.

   Solution: By the quotient rule,

$$f'(x) = \frac{-\sin x(2+\sin x) - \cos x(\cos x)}{(2+\sin x)^2} = \frac{-2\sin x - 1}{(2+\sin x)^2}.$$

   (b) Find the critical points of f.

   Solution: The only zero of $f'(x)$ occurs when $\sin x = -1/2$ which is true when $x = \pi + \pi/6 = 7\pi/6$ and $x = 2\pi - \pi/6 = 11\pi/6$

   (c) Identify each critical point as a location where a max, a min, or neither occurs.

   Solution: Use the test interval technique with f' on the interval $[0, 2\pi]$, noting that $f'(\pi) < 0, f'(3\pi/2) > 0$, and $f'(2\pi^-) < 0$. Thus f has a relative minimum at $x = 7\pi/6 \approx 3.665$ and a relative maximum at $x = 11\pi/6 \approx 5.759$.

   (d) Find the absolute maximum and absolute minimum of f.

   Solution: Comparing the value of f at the endpoints and the two critical points, we find $f(0) = 1/2, f(2\pi) = 1/2,\ f(7\pi/6) = -1/\sqrt{3}$, and $f(11\pi/6) = 1/\sqrt{3} \approx 0.5773$, so the min occurs at $x = 7\pi/6$ and the max occurs at $x = 11\pi/6$.

4. (15 points) The mean value theorem (MVT) states that if f is differentiable over $[a, b]$, then there is a number c in (a, b) such that $f'(c)$ is the slope of the line joining $(a, f(a))$ and $(b, f(b))$.

   (a) Does the MVT apply to the function $f(x) = x \ln x$ on the interval $[1, e]$.

   Solution: Yes, MVT applies because f is differentiable on $(1, e)$ and continuous at both 1 and e.

   (b) If not tell why. If so, find the number c.

   Solution: Note that $\frac{f(e)-f(1)}{e-1} = \frac{e \ln e - 1 \ln 1}{e-1} = \frac{e}{e-1}$. So we need to find a number c in the interval satisfying $f'(c) = \frac{e}{e-1}$. Since $f'(x) = \ln x + 1$, we need to find c such that $\ln c + 1 = \frac{e}{e-1}$. This gives $\ln c = \frac{e}{e-1} - 1 = \frac{1}{e-1}$. We can interpret $\ln c$ to get $c = e^{\frac{1}{e-1}} \approx 1.7895$.

5. (20 points) Suppose f is a differentiable function and suppose f'' is given by

$$f''(x) = \frac{(x^2-4)(x+5)}{(x+2)(x+1)}.$$

Find the intervals over which f is concave up. No credit for calculator solutions.

Solution: First, reduce f'' by removing the common factors to get $f''(x) = \frac{(x-2)(x+5)}{x+1}$. Of course, we have slightly enlarged the domain of f'' in doing this. There are three branch points, $x = 2$, $x = -5$ and $x = -1$ to consider. Use the test interval technique to solve the inequality $f''(x) > 0$. I used the test points $-6, -2, 0$, and 3 and found that $f''(-6) < 0$, $f''(-2) > 0$, $f''(0) < 0$, and $f''(3) > 0$. Thus f is concave upwards on the two intervals $(-5, -1)$ and $(2, \infty)$.

6. (20 points) Evaluate each of the following limits:

(a) $\lim\limits_{x \to -1} \dfrac{x^2-1}{x+1}$

Solution: One way to evaluate this limit is by factoring. Another is to apply L'Hospital's Rule and differentiate both numerator and denominator. $\lim\limits_{x \to -1} \dfrac{x^2-1}{x+1} = \lim\limits_{x \to -1} \dfrac{2x}{1} = -2$

(b) $\lim\limits_{x \to 0} \dfrac{x + \tan x}{\sin x}$

Solution: Apply L'Hospital's Rule and differentiate both numerator and denominator to get $\lim\limits_{x \to -1} \dfrac{x + \tan x}{\sin x} =$
$\lim\limits_{x \to -1} \dfrac{1 + \sec^2 x}{\cos x} = 2$

(c) $\lim\limits_{x \to \infty} x^3 e^{-x^2}$

Solution: Apply L'Hospital's Rule repeatedly to get

$$\begin{aligned}\lim_{x\to\infty} x^3e^{-x^2} &= \lim_{x\to\infty} x^3/e^{x^2} \\ &= \lim_{x\to\infty} 3x^2/2xe^{x^2} \\ &= \lim_{x\to\infty} 6x/(2e^{x^2}+4x^2e^{x^2}) \\ &= \lim_{x\to\infty} 6/(4xe^{x^2}+8xe^{x^2}+8x^3e^{x^2}) = 0\end{aligned}$$

(d) $\lim_{x\to 0}(1-2x)^{\frac{1}{x}}$

Solution: Take ln of the expression to get a fraction. Note that

$$\lim_{x\to 0}\ln((1-2x)^{\frac{1}{x}}) = \lim_{x\to 0}\frac{1}{x}\ln((1-2x) = \lim_{x\to 0}\frac{\ln((1-2x)}{x}.$$

Next apply L'hospital's rule to get $\lim_{x\to 0}\frac{\ln((1-2x)}{x} = \lim_{x\to 0}\frac{-2/(1-2x)}{1} = -2$. Therefore, the original limit must be e^{-2}.