September 27, 2004 **Name** ____________________________

On all the following questions, **show your work.** There are 145 points available on this test. Do not try to do all the problems. Try to find four or five that you you can do well.

1. (15 points) Let $f(x) = 1/x$ for all $x > 0$, and let $[a, b] = [1, 3]$.

   (a) Let $n = 4$ and use left endpoints for sample points to find the approximating sum. That is, compute L_4.

   Solution: The sum is $\Delta x(f(x_0) + f(x_1) + f(x_2) + f(x_3))$
   $= \frac{1}{2}(f(1) + f(1.5) + f(2) + f(2.5)) = (1 + 2/3 + 1/2 + 2/5)/2 = 77/60$.

   (b) Find the n^{th} approximating sum, also using left endpoints. In other words, find an expression for L_n. You need not evaluate the limit as $n \to \infty$.

   Solution: $\lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \frac{1}{1+\frac{2(i-1)}{n}}$.

2. (30 points) Use the evaluation theorem, etc. to find each of the definite integrals below.

   (a) $\displaystyle\int_0^{\ln 3} 2e^{2x} dx$

   Solution: By substitution with $u = 2x$, $\displaystyle\int_0^{\ln 3} 2e^{2x} dx = e^{2x}\big|_0^{\ln 3} = 9 - 1 = 8$.

   (b) $\displaystyle\int_1^6 \frac{x^2 - 3x + 5}{x^2}\, dx$

   Solution: Break up the integrand into three functions to get

$$\begin{aligned}
\int_1^6 \frac{x^2 - 3x + 5}{x^2}\, dx &= \int_1^6 \frac{x^2}{x^2}\, dx + \int_1^6 \frac{-3x}{x^2}\, dx + \int_1^6 \frac{5}{x^2}\, dx \\
&= x - 3\ln x - 5x^{-1}\big|_1^6 \\
&= 6 - 3\ln 6 - 5/6 - (1 - 5) = 55/6 - 3\ln 6 \approx 3.79
\end{aligned}$$

   (c) $\displaystyle\int_0^{\pi/4} \sec^2 x\, dx$

   Solution: Recall that $\frac{d}{dx}\tan x = \sec^2 x$, so $\displaystyle\int_0^{\pi/4} \sec^2 x\, dx = \tan x\big|_0^{\pi/4} = \tan \pi/4 - \tan 0 = 1$.

(d) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

Solution: $\sin^{-1}|_0^1 = \pi/2 \approx 1.57$. Alternatively, let $x = \sin\theta$. Then $dx = \cos\theta d\theta$, $1 - x^2 = 1 - \sin^2 x = \cos^2 x$ and the integral reduces to $\displaystyle\int_0^1 \frac{\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta = \int 1\,d\theta = \arcsin x|_0^1 = \arcsin 1 - \arcsin 0 = \pi/2$.

(e) $\displaystyle\int_0^1 \frac{x}{1+x^4}\,dx$

Solution: Let $u = x^2$. Then $du/2 = x\,dx$ and our integral can be written $1/2 \cdot \displaystyle\int_0^1 \frac{1}{1+u^2}\,du$ which is just $.5\arctan u^2|_0^1 = .5(\pi/4 - 0) = \pi/8$.

3. (15 points) Evaluate $\displaystyle\int_0^\infty 1/x^2\,dx$.

Solution: This integral is doubly improper. There is a vertical asymptote at $x = 0$ besides being an infinite integral. First, define the integral as $\displaystyle\int_0^1 1/x^2\,dx + \int_1^\infty 1/x^2\,dx$. The second converges, but the first does not, so the integral diverges.

4. (20 points) Let $g(x) = \displaystyle\int_0^x t^2 - 6t + 5\,dt$.

(a) Over what intervals is g increasing?

Solution: To see where $g(x)$ is increasing, take the derivative. Since $g'(x) = x^2 - 6x + 5$ by the fundamental theorem, it follows that g is increasing precisely when $x^2 - 6x + 5 = (x-5)(x-1) > 0$. This occurs when $x < 1$ and when $x > 5$.

(b) Over what intervals is g concave upwards?

Solution: To establish concave upwardness, we want $g''(x) > 0$. This happens when $2x - 6 > 0$, or equivalently, $x > 3$.

(c) What is the maximum value of g over the interval $[0, 8]$?

Solution: To maximize the function, examine its value at a) the critical points $x = 1$ and $x = 5$ and b) the endpoints $x = 0$ and $x = 8$. Note that $g(8) = 18\frac{2}{3}$ which is bigger than $g(0), g(1)$, and $g(5)$.

5. (20 points) Suppose f is defined by:

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x \le 1 \\ x-1 & \text{if } 1 < x \le 2 \\ 3-x & \text{if } 2 < x \le 3 \\ 0 & \text{if } 3 < x \end{cases}$$

Let $g(x) = \int_0^x f(t)dt$.

(a) Find an expression for $g(x)$ similar to the one for $f(x)$.

Solution:

$$g(x) = \begin{cases} 0 & \text{if } 0 \le x \le 1 \\ x^2/2 - x + 1/2 & \text{if } 1 < x \le 2 \\ 3x - x^2/2 - 7/2 & \text{if } 2 < x \le 3 \\ 1 & \text{if } 3 < x \end{cases}$$

For example, if $2 < x < 3$, then

$$\begin{aligned} g(x) &= \int_0^x f(t)dt \\ &= \int_0^1 f(t)dt + \int_1^2 f(t)dt + \int_2^x f(t)dt \\ &= 0 + 1/2 + \int_2^x 3 - t\ dt \\ &= 1/2 + 3t - t^2/2|_2^x \\ &= 1/2 + 3x - x^2/2 - 4 \end{aligned}$$

(b) Sketch the graphs of g and f.

Solution: The graph of f is just the x-axis except for a segment from $(1,0)$ to $(2,1)$ and another segment from $(2,1)$ to $(3,0)$. The graph of g looks like the parabola $x^2 - x$ on $[1,2]$ and the parabola $3x - x^2/2$ on $[2,3]$. Notice that g is continuous.

(c) Compute $g'(x)$.

Solution: $g'(x) = \begin{cases} 0 & \text{if } 0 \le x \le 1 \\ x-1 & \text{if } 1 < x \le 2 \\ 3-x & \text{if } 2 < x \le 3 \\ 0 & \text{if } 3 < x \end{cases}$

6. (15 points) Evaluate $\displaystyle\int_0^1 \ln x \, dx$.

   Solution: This is example 8, page 434 in the text. An antiderivative of $\ln x$ is obtainable by integration by parts, $x \ln x - x$ works. Next note that the integral is improper, so we must compute

$$\lim_{t\to 0^+} \int_t^1 \ln x \, dx = \lim_{t\to 0^+} x \ln x - x|_t^1 = -1 + \lim_{t\to 0^+} \frac{\ln t}{1/t} = -1 + \lim_{t\to 0^+} \frac{1/t}{-1/t^2} = -1 + 0,$$

   where L'Hospital's rule was applied to get the last limit.

7. (15 points) Let $f(x) = \displaystyle\int_0^{x^2} \frac{t}{t^3 - 2}\, dt$. Then $f'(x) = \frac{2x^3}{x^6-2}$. Explain why this is the case. How does the chain rule play a part here? What functions are being composed?

   Solution: The key here is to define another function, say $h(x)$ as follows. $h(x) = \int_0^x \frac{t}{t^3-2}\, dt$. Now the fundamental theorem applies nicely to h to tell us that $h'(x) = \frac{x}{x^3-2}$. Notice also that $f(x) = h(x^2)$, so the chain rule applies to tell us how to differentiate f: $f'(x) = h'(x^2) \cdot 2x = \frac{x^2}{(x^2)^3-2} \cdot 2x$.

8. (15 points) Consider the integral $\displaystyle\int_{-2}^{3} 1/x\, dx$.

   (a) It is tempting to evaluate this integral by antidifferentiating $f(x) = 1/x$, getting $F(x) = \ln|x|$, and then to measuring the growth of $F(x)$ over the interval $[-2, 3]$ to get $\ln|3| - \ln|-2| = \ln 3 - \ln 2 = \ln(3/2)$. Explain why this is wrong.

   Solution: The integral can't be evaluated this way because the evaluation theorem requires that the antiderivative be valid over the entire interval, but this one isn't valid at 0. It is valid on BOTH sides of zero.

   (b) Is there are reasonable approach to this problem?

   Solution: The solution is to build two improper integrals, one from -2 to 0 and the other from 0 to 3. Neither of these integrals converges, however.