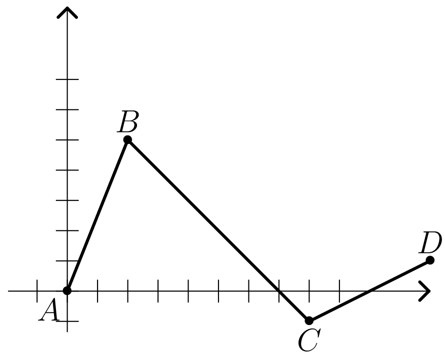
**February 18, 2005** **Name** ____________________________

On all the following questions, **show your work.** There are 130 points available on this test. Do not try to do all the problems. Try to find four or five that you you can do well.

1. (15 points) You are given the four points in the plane $A = (0, 0), B(2, 5), C = (8, -1)$ and $D = (12, 1)$. The graph of the function $f$ consists of the three line segments $AB$, $BC$ and $CD$. Find the integral by interpreting the integral in terms of sums and/or differences of areas of elementary figures.




**Solution:** Notice that the segment from $B$ to $C$ has slope $-1$ and that it crosses the $x$-axis at $(0, 7)$. The triangle with vertices $A, B$ and $(0, 7)$ has area $35/2$, and the triangle with vertices $(0, 7), (0, 8), C$ has area $1/2$. This subtracts from $35/2$ because it lies below the $x$-axis. The rest of the integral (from $x = 8$ to $x = 12$) is zero, so the answer we want it $35/2 - 1/2 = 34/2 = 17$.

2. (15 points) Let $f(x) = 1/x$ for all $x > 0$, and let $[a, b] = [1, 3]$.

   (a) Let $n = 4$ and use right endpoints for sample points to find the approximating sum. That is, compute $R_4$.

   **Solution:** The sum is $\Delta x(f(x_1) + f(x_2) + f(x_3) + f(x_4))$
   $= \frac{1}{2}(f(1.5) + f(2) + f(2.5) + f(3)) = (2/3+1/2+2/5+1/3)/2 = 57/60 = 19/20$.

   (b) Find the $n^{\text{th}}$ approximating sum, also using right endpoints. In other words, find an expression for $R_n$. You need not evaluate the limit as $n \to \infty$.

   **Solution:** $\displaystyle\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} \frac{n}{n+2i} = \lim_{n\to\infty} 2\sum_{i=1}^{n} \frac{1}{n+2i}$.

3. (20 points) Find the following indefinite integrals.

   (a) $\int \frac{2x}{x^2+4} dx$

   **Solution:** By substitution with $u = x^2+4$, $\int \frac{du}{u} = \ln u = \ln(x^2+4)+C$.

   (b) $\int \frac{1}{x^2+4}\ dx$

   **Solution:** Use the substitution $x = 2\tan\theta$ to get $\frac{1}{2}\tan^{-1}(x/2) + C$.

   (c) $\int \cos^3 x\ \ \sin^2 x\ dx$

   **Solution:** Rewrite the integral as $\int \cos x \cos^2 x\ \ \sin^2 x\ dx = \int \cos x(1 - \sin^2 x)\sin^2 x\ dx$. Then use the substitution $u = \sin x, du = \cos x dx$ to get the final answer $\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$.

4. (20 points) Let $g(x) = \int_0^x (4-t)(2+t)(5+t)\ dt$. Over what intervals is $g$ increasing?

   **Solution:** To see where $g(x)$ is increasing, take the derivative. Since $g'(x) = (4-x)(2+x)(5+x)$ by the fundamental theorem, it follows that $g$ is increasing precisely when $(4-x)(2+x)(5+x) > 0$. This occurs when $x < -5$ and when $-2 < x < 4$.

5. (30 points) Use the evaluation theorem as needed to find each of the definite and improper integrals below.

   (a) $\displaystyle\int_3^4 \frac{x+1}{x^2-4}\,dx$

   **Solution:** Use partial fractions to decompose the integrand as follows: $\frac{x+1}{x^2-4} = \frac{x+1}{(x-2)(x+2)} = A/(x-2) + B/(x+2)$ Solve for $A$ and $B$ to get $\displaystyle\int_3^4 \frac{1}{4(x-2)} + \frac{3}{4(x+2)}\,dx$. Then anti-differentiate to get $(1/4)\ln(x-2) + (3/4)\ln(x+2)|_3^4 = (1/4)\ln 6/5 + (3/4)\ln 2 = \ln(48/5)/4 \approx 0.5654$. No credit for this problem without the antiderivative.

   (b) $\displaystyle\int_0^2 xe^{x^2}\,dx$

   **Solution:** Let $u = x^2$. Then $du/2 = x\,dx$ and our integral can be written $\displaystyle\frac{1}{2}\int_0^1 e^u\,du$ which is just $\frac{1}{2}(e^4-1)$. No credit for this problem without the antiderivative.

   (c) $\displaystyle\int_0^1 x^2(x-1)^8\,dx$

   **Solution:** Let $u = x-1$. Then $du = dx$, $x = u+1$, $x^2 = u^2+2u+1$ and our integral can be written $\displaystyle\int_0^1 (u^2+2u+1)u^8\,du$ which is just $(u^{11}/11 + 2u^{10}/10 + u^9/9)|_0^1 = 20/99 - 20/100 = 0.00\overline{202}$. No credit for this problem without the antiderivative.

   (d) $\displaystyle\int_e^\infty (x\ln x)^{-1}\,dx$

   **Solution:** Use substitution with $u = \ln x$ and $du = x^{-1}dx$. Then $\displaystyle\int_e^\infty (x\ln x)^{-1}\,dx = \lim_{t\to\infty}\int_e^t (x\ln x)^{-1}\,dx = \lim_{t\to\infty}\int_1^{\ln t} 1/u\,du = \lim_{t\to\infty} \ln u|_1^{\ln t} = \lim_{t\to\infty} \ln\ln t - \ln 1 = \lim_{t\to\infty} \ln\ln t$, which diverges. At most 2 points credit for this problem without the antiderivative.

   (e) $\displaystyle\int_1^\infty 1/x^2\,dx.$

   **Solution:** This integral is improper. First, define the integral as $\displaystyle\lim_{t\to\infty}\int_1^t 1/x^2\,dx$. This converges to the value 1. No credit for this problem without the antiderivative.

6. (15 points) Let $f(x) = \displaystyle\int_0^{\sqrt{x}} \frac{t^2}{t^4+2}\,dt$. Then $f'(x) = \frac{\sqrt{x}}{2x^2+4}$. Explain why this is the case. How does the chain rule play a part here? What functions are being composed?

   **Solution:** The key here is to define another function, say $h(x)$ as follows. $h(x) = \int_0^x \frac{t^2}{t^4+2}\,dt$. Now the fundamental theorem applies nicely to $h$ to tell us that $h'(x) = \frac{x^2}{x^4+2}$. Notice also that $f(x) = h(\sqrt{x})$, so the chain rule applies to tell us how to differentiate $f$: $f'(x) = h'(\sqrt{x}) \cdot \frac{d}{dx}\sqrt{x} = \frac{(\sqrt{x})^2}{(\sqrt{x})^4)+2} \cdot x^{-1/2}/2 = \frac{\sqrt{x}}{2x^2+4}$.

7. (15 points) Consider the integral $\displaystyle\int_{-2}^{3} 1/x\,dx$.

   (a) It is tempting to evaluate this integral by antidifferentiating $f(x) = 1/x$, getting $F(x) = \ln|x|$, and then to measuring the growth of $F(x)$ over the interval $[-2,3]$ to get $\ln|3| - \ln|-2| = \ln 3 - \ln 2 = \ln(3/2)$. Explain why this is wrong.

   **Solution:** The integral can't be evaluated this way because the evaluation theorem requires that the antiderivative be valid over the entire interval, but this one isn't valid at 0. It is valid on BOTH sides of zero.

   (b) Is there are reasonable approach to this problem?

   **Solution:** The solution is to build two improper integrals, one from $-2$ to 0 and the other from 0 to 3. Neither of these integrals converges, however.