September 27, 2005 **Name** ______________________

On all the following questions, **show your work.** There are 144 points available on this test. Do not try to do all the problems. Try to find four or five that you you can do well.

1. (20 points) Let $f(x) = 1/x$ for all $x > 0$, and let $[a, b] = [2, 8]$.

   (a) (5) Let $n = 3$ and use left endpoints for sample points to find the approximating sum. That is, compute L_3.

   Solution: Note that $\Delta x = \frac{8-2}{3} = 2$, so the sum is $\Delta x(f(x_1) + f(x_2) + f(x_3) + f(x_4))$
   $= 2\,(f(2) + f(4) + f(6)) = 2(1/2 + 1/4 + 1/6) = 11/6.$

   (b) (7) Find the n^{th} approximating sum, also using left endpoints. In other words, find an expression for L_n. You need not evaluate the limit as $n \to \infty$.

   Solution: Note that $\Delta x = \frac{8-2}{n} = 6/n$. Thus,

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_{i-1})\Delta x &= \lim_{n\to\infty}\frac{6}{n}\sum_{i=1}^{n} f(x_{i-1}) \\ &= \lim_{n\to\infty}\frac{6}{n}\sum_{i=1}^{n} f(2 + 6(i-1)/n) \\ &= \lim_{n\to\infty}\frac{6}{n}\sum_{i=1}^{n}\frac{1}{(2 + 6(i-1)/n)}.\end{aligned}$$

   (c) (8) Use the midpoint rule to approximate $\int_2^8 1/x\;dx$. Compare the two numbers M_3 and $\int_2^8 1/x\;dx$.

   Solution: $M_3 = \Delta x(f(3)+f(5)+f(7) = 2\,(1/3 + 1/5 + 1/7)) \approx 1.35238$. On the other hand, $\int_2^8 1/x\;dx = \ln 8 - \ln 2 = 3\ln 2 - \ln 2 = 2\ln 2 \approx 1.38629$. Therefore M_3 is an underestimate by more than 3/100.

2. (24 points) Find the following indefinite integrals.

   (a) $\displaystyle\int \frac{(x-1)^2}{x^2+1}dx$

   Solution: Since $(x-1)^2 = x^2 - 2x + 1$, the integral breaks nicely into two, $\displaystyle\int \frac{(x-1)^2}{x^2+1}dx = \int \frac{x^2+1}{x^2+1}dx - \int \frac{2x}{x^2+1}dx = x - \ln(x^2+1) + C$.

   (b) $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\ dx$

   Solution: Let $x = 3\sin\theta$. Then $dx = 3\cos\theta d\theta$ and $9 - x^2 = 9 - 9\sin^2\theta = 9\cos^2\theta$ which is nonnegative for $0 \le \theta \le \pi/2$. It follows that $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\ dx = \int 1 d\theta = \theta + C = \sin^{-1}(x/3) + C$.

   (c) $\displaystyle\int \frac{d}{dx}(x-3)(x^2-1)\ dx$

   Solution: On course the antiderivative of the derivative is just the function, so $\displaystyle\int \frac{d}{dx}(x-3)(x^2-1)\ dx = (x-3)(x^2-1) + C$

3. (40 points) Use the evaluation theorem as needed to find each of the definite and improper integrals below.

   (a) $\displaystyle\int_0^{\pi/2} \cos x \cos(\sin x)\ dx$

   Solution: Let $u = \sin x$. Then $du = \cos x\ dx$ and $\displaystyle\int \cos x \cos(\sin x)\ dx = \sin u = \sin(\sin(x))$. Therefore, $\displaystyle\int_0^{\pi/2} \cos x \cos(\sin x)\ dx = \sin(\sin(x))|_0^{\pi/2} = \sin 1 - \sin 0 \approx 0.84147$. Be sure the calculator is in radian mode for this calculation.

(b) $\displaystyle\int_3^4 \frac{x+1}{x^2-4}\,dx$

Solution: Use partial fractions to decompose the integrand as follows: $\frac{x+1}{x^2-4} = \frac{x+1}{(x-2)(x+2)} = A/(x-2) + B/(x+2)$ Solve for A and B to get $\displaystyle\int_3^4 \frac{3}{4(x-2)} + \frac{1}{4(x+2)}\,dx$. Then anti-differentiate to get $(3/4)\ln(x-2) + (1/4)\ln(x+2)|_3^4 = (3/4)\ln 2 + (1/4)\ln 6 - (1/4)\ln 5 \approx 0.56544$.

(c) $\displaystyle\int_e^\infty (x\ln x)^{-1}\,dx$

Solution: Use the substitution $u = \ln x$. Then $\int (x\ln x)^{-1}\,dx = \ln|u| = \ln(|\ln|x||)$, so $\int_e^\infty (x\ln x)^{-1}\,dx = \lim_{t\to\infty}\ln(\ln x)|_e^t = \lim_{t\to\infty}\ln(\ln t)$, which diverges because $\ln t$ is unbounded.

(d) $\displaystyle\int_0^2 xe^{x^2}\,dx$

Solution: Let $u = x^2$. Then $du/2 = x\,dx$ and our integral can be written $\displaystyle\frac{1}{2}\int_0^4 e^u\,du$ which is just $\frac{1}{2}(e^4-1) \approx 26.80$.

(e) $\displaystyle\int_0^1 x^2(x-2)^8\,dx$

Solution: Let $u = x-2$. Then $du = dx,\ x = u+2,\ x^2 = u^2+4u+4$ and our integral can be written $\displaystyle\int_0^1 (u^2+4u+4)u^8\,du$ which is just $(u^{11}/11 + 4u^{10}/10 + 4u^9/9)|_{-2}^{-1} \approx 4.00202$.

4. (20 points) Consider the integral $\int_{-2}^{3} 1/x \; dx$.

   (a) Explain why this integral is not defined by the usual definition of integral as a limit of Riemann sums as the number of subintervals n approaches ∞.

   Solution: The function $1/x$ is unbounded on $[-2, 3]$. Therefore the Riemann sums are also unbounded, and $lim_{n\to\infty} R_n$ does not exist.

   (b) It is tempting to evaluate this integral by antidifferentiating $f(x) = 1/x$, getting $F(x) = \ln|x|$, and then to measuring the growth of $F(x)$ over the interval $[-2, 3]$ to get $\ln|3| - \ln|-2| = \ln 3 - \ln 2 = \ln(3/2)$. Explain why this is wrong.

   Solution: The integral can't be evaluated this way because the evaluation theorem requires that the antiderivative be valid over the entire interval, but this one isn't valid at 0 since $1/x$ is not defined there. It is valid on both sides of zero.

   (c) Is there are reasonable approach to this problem? What is it?

   Solution: The solution is to build two improper integrals, one from -2 to 0 and the other from 0 to 3. Thus, $\int_{-2}^{3} 1/x \; dx = \lim_{t\to 0^-} \left(\int_{-2}^{t} 1/x \; dx\right) + \lim_{t\to 0^+} \left(\int_{t}^{3} 1/x \; dx\right)$. Neither of these integrals converges, however.

5. (25 points) Let $g(x) = \int_0^{2x^2} (t-2)(t-8)\ dt$.

   (a) Find $g'(x)$.
   Solution:

$$\begin{aligned} g'(x) &= (2x^2-2)(2x^2-8)\cdot 4x \\ &= 16x(x^2-1)(x^2-4) \\ &= 16x(x-1)(x+1)(x-2)(x+2) \end{aligned}$$

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   (b) Find the critical points of g. IE, find the zeros of g'.
   Solution: The zeros of $g'(x)$ are $-2, -1, 0, 1$, and 2.

   (c) Compute $g'(-3/2), g'(-1/2), g'(1/2)$, and $g'(3/2)$.
   Solution: $g'(-3/2) = 105/2 > 0$. Similarly $g'(-1/2) < 0$, $g'(1/2) > 0$, and $g'(3/2) < 0$.

   (d) Recall the great theorem in calculus 1 that tells you when a differentiable function is increasing: If $f'(x) > 0$ at every point of (a, b), then f is increasing over (a, b). Use this theorem and the Test Interval Technique to find the intervals over which g is increasing. Recall that $g'(x)$ must be factored completely to apply the Test Interval Technique.

   Solution: g is increasing on each of the intervals $[-2, -1], [0, 1]$, and $[2, \infty)$.

6. (15 points) Use the substitution $x = \sec\theta$ to compute $\int \frac{\sqrt{x^2-1}}{x^4}\ dx$. Show the triangle you use to find $\sin\theta$, etc. You may find useful the formula $\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$.

   Solution: Note that $x = \sec\theta$, $x^2 = \sec^2\theta$, $\sqrt{x^2-1} = \sqrt{\tan^2\theta} = \tan\theta$, and $dx = \sec\theta\tan\theta d\theta$. Thus $\int \frac{\sqrt{x^2-1}}{x^4} dx = \int \frac{\tan\theta\sec\theta\tan\theta\ d\theta}{\sec^3\theta\sec\theta}$, which reduces to $\int \sin^2\theta\cos\theta\ d\theta$. Now let $u = \sin\theta$. The integral is thus $\int u^2\ du = u^3/3 = \sin^3\theta/3 = \frac{\sqrt{(x^2-1)^3}}{x^3} + C$. The triangle in question has sides of length 1 and $\sqrt{x^2-1}$ and hypotenuse of x.