February 23, 2006   Name _____

On all the following questions, **show your work.** There are 143 points available on this test. Do not try to do all the problems. Try to find four or five that you you can do well.

1. (10 points) Suppose $\int_{-8}^{-5} f(x)dx = 2$, $\int_{-8}^{-7} f(x)dx = 10$, $\int_{-6}^{-5} f(x)dx = 8$.

   (a) Find $\int_{-7}^{-6} f(x)dx =$

   Solution: Using property 5 of integrals, note that $\int_{-7}^{-6} f(x)dx = \int_{-7}^{-8} f(x)dx +$
   $\int_{-8}^{-5} f(x)dx + \int_{-5}^{-6} f(x)dx = -10 + 2 + (-8) = -16$.

   (b) $\int_{-6}^{-7}(2f(x) - 10)dx =$

   Solution: Using what we learned above and the linearity of the integral,
   $\int_{-6}^{-7}(2f(x) - 10)dx = -(2(-16) + 10(-6 - (-7))) = 42$.

2. (15 points) Given
$$f(x) = \int_0^x \frac{t^2 - 4}{1 + \cos^2(t)}dt$$
   At what value of x does the local max of $f(x)$ occur?

   Solution: By FTC, the derivative of f is $f'(x) = \frac{x^2-4}{1+\cos^2(x)}$, which is positive on $(-\infty, -2) \cup (2, \infty)$. So f is increasing to the left of -2 and decreasing to the right of -2. Therefore it has a local (relative) maximum at -2. The other extremum is at $x = 2$ where f has a local minimum.

3. (24 points) Find the following indefinite integrals.

(a) $\displaystyle\int \cos^3\theta \sin^3\theta\ d\theta$

Solution: Let $u = \sin\theta$, then $du = \cos\theta d\theta$. Replace $\cos^2\theta$ with $1 - \sin^2\theta$ and the integral becomes $\displaystyle\int u^3 - u^5\ du$. Thus $\displaystyle\int \cos^3\theta\sin^3\theta\ d\theta = \frac{\sin^4\theta}{4} - \frac{\sin^6\theta}{6} + C$.

(b) $\displaystyle\int (x-1)^2 dx$

Solution: Since $(x-1)^2 = x^2 - 2x + 1$, the integral is simply $\dfrac{x^3}{3} - x^2 + x + C$.

(c) $\displaystyle\int \frac{2x}{x^2+1} dx$

Solution: Let $u = x^2 + 1$. Then $du = 2xdx$ and $\displaystyle\int \frac{2x}{x^2+1} dx = \int 1/u\ du = \ln\mid u \mid + C = \ln(x^2+1) + C$.

(d) $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\ dx$

Solution: Let $x = 2\sin\theta$. Then $dx = 2\cos\theta d\theta$ and $4 - x^2 = 4 - 4\sin^2\theta = 4\cos^2\theta$ which is nonnegative for $0 \le \theta \le \pi/2$. It follows that $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\ dx = \int 1 d\theta = \theta + C = \sin^{-1}(x/2) + C$.

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4. (64 points) Use the evaluation theorem as needed to find each of the definite and improper integrals below. Each improper integral must be identified as such to get credit.

(a) $\int_0^2 \frac{d}{dx}[(x^2 - 3)(x^3 - 1)]\ dx$

Solution: On course the integral of the derivative is just the growth of the function, so $\int_0^2 \frac{d}{dx}(x^2 - 3)(x^3 - 1)\ dx = (x^2 - 3)(x^3 - 1)|_0^2 = (4 - 3)(8 - 1) - (-3)(-1) = 7 - 3 = 4$.

(b) $\int_4^9 \frac{9}{\sqrt{x}}dx$

Solution: Just use the power rule to get $9 \cdot 2 \cdot x^{1/2}|_4^9 = 18(3 - 2) = 18$.

(c) $\int_0^{\pi/2} \cos x \cos(\sin x)\ dx$

Solution: Let $u = \sin x$. Then $du = \cos x\ dx$. Therefore, $\int_0^{\pi/2} \cos x \cos(\sin x)\ dx = \int \cos x \cos(u)\ du = \sin(\sin(x))|_0^{\pi/2} = \sin 1 - \sin 0 \approx 0.84147$. Be sure the calculator is in radian mode for this calculation.

(d) $\int_3^4 \frac{x - 1}{x^2 - 4}\ dx$

Solution: Use partial fractions to decompose the integrand as follows: $\frac{x - 1}{x^2 - 4} = \frac{x - 1}{(x - 2)(x + 2)} = A/(x - 2) + B/(x + 2)$. Solve for A and B to get $A = 1/4$ and $B = 3/4$. Therefore we have $\int_3^4 \frac{1}{4(x - 2)} + \frac{3}{4(x + 2)}\ dx$. Then anti-differentiate to get $(1/4)\ln(x - 2) + (3/4)\ln(x + 2)|_3^4 = (1/4)\ln 2 + (3/4)\ln 6 - (3/4)\ln 5 \approx 0.3100$.

(e) $\displaystyle\int_e^\infty (x \ln x)^{-1} \, dx$

Solution: Use the substitution $u = \ln x$. Then $\int (x \ln x)^{-1} \, dx = \ln|u| = \ln(|\ln|x||)$, so $\int_e^\infty (x \ln x)^{-1} \, dx = \lim_{t\to\infty} \ln(\ln x)|_e^t = \lim_{t\to\infty} \ln(\ln t)$, which diverges because $\ln t$ is unbounded.

(f) $\displaystyle\int_1^4 |x^3 - 6x^2 + 11x - 6| \, dx$. Note that $f(x) = x^3 - 6x^2 + 11x - 6$ factors into $(x-1)(x-2)(x-3)$.

Solution: Break the integral into three integrals, $\int_1^2, \int_2^3, \int_3^4$. Note that $f(x)$ is positive over the first and last of these and negative over the second one. Therefore, $\displaystyle\int_1^4 |x^3 - 6x^2 + 11x - 6| \, dx = \int_1^2 f(x) - \int_2^3 f(x) + \int_3^4 f(x) = F(2) - F(1) - (F(3) - F(2)) + F(4) - F(3) = 0.25 + 0.25 + 2.25 = 2.75$, where $F(x) = x^4/4 - 2x^3 + 11x^2/2 - 6x$.

(g) $\displaystyle\int_0^1 x(x-2)^9 \, dx$

Solution: Let $u = x - 2$. Then $du = dx$, $x = u + 2$ and our integral can be written $\displaystyle\int_0^1 (u+2)u^9 \, du$ which is just $(u^{11}/11 - 2u^{10}|_{-2}^{-1} = \dfrac{2^2 - 10 - 2^{11}}{110} \approx -18.509$.

(h) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}} \, dx$

Solution: This integral is improper since the integrand has a vertical asymptote at $x = 0$. Thus we have $\displaystyle\int_0^1 \frac{1}{\sqrt{x}} \, dx = \lim_{t\to 0^+} \int_0^t \frac{1}{\sqrt{x}} \, dx = \lim_{t\to 0^+} 2x^{1/2}|_t^1 = \lim_{t\to 0^+} 2 - 2t^{1/2} = 2$.

5. (15 points) Construct a triangle with an acute angle θ such that $\tan\theta = x/2$. Then compute each of the following in terms of x.

   (a) $\sin\theta$

   Solution: $\sin\theta = \dfrac{x}{\sqrt{x^2 + 2^2}}$.

   (b) $\sin(2\theta)$

   Solution: $\sin(2\theta) = 2\sin\theta\cos\theta = 2\dfrac{x}{\sqrt{x^2 + 2^2}} \cdot \dfrac{2}{\sqrt{x^2 + 2^2}} = \dfrac{4x}{x^2 + 2^2}$.

   (c) $\csc\theta$

   Solution: $\csc\theta = \dfrac{\sqrt{x^2 + 2^2}}{x}$.

6. (15 points) For each integral below, use the substitution θ such that $x = 2\tan\theta$ to find an equivalent $d\theta$ integral. Do not evaluate.

   (a) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4 + x^2}}\, dx$

   Solution: $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4 + x^2}}\, dx = \int_0^{\pi/4} \frac{4\tan^2\theta \cdot 2\sec^2\theta\ d\theta}{2\sec\theta}$.

   (b) $\displaystyle\int \frac{1}{4 + x^2}\, dx$

   Solution: $\displaystyle\int \frac{1}{4 + x^2}\, dx = \frac{1}{2}\int 1\,d\theta$.

   (c) $\displaystyle\int x\sqrt{4 + x^2}\, dx$

   Solution: $\displaystyle\int x\sqrt{4 + x^2}\, dx = \int 4\tan\theta \cdot 2\sec\theta \cdot 2\sec^2\theta\, d\theta = 8\int \tan\theta \cdot \sec^3\theta\, d\theta$.