October 29, 2004 **Name** ____________________________

On all the following questions, **show your work.** There are 125 points available on this test. Do not try to do all the problems. Try to find four or five that you you can do well.

1. (15 points) Find the area enclosed by the curves $f(x) = x^2 + 1$ and $g(x) = 16 + x - x^2$.

   Solution: First find the places where the graphs intersect by solving $x^2+1 = 16 + x - x^2$. You get $x = -5/2$ and $x = 3$. Then integrate the difference over that interval.
   $\int_{-5/2}^{3} -2x^2 + x + 15\ dx = (-2x^3/3 + x^2/2 + 15x)|_{-5/2}^{3} = 1331/24 \approx 55.45$

2. (15 points) Find the average value of the function $f(x) = 2x(x^2 - 1)^{3/2}$ over the interval $[1, \sqrt{5}]$. Verify that there exists a number c, $1 \le c \le \sqrt{5}$ such that $f(c) = f_{ave}$. In other words verify that the conclusion of the mean value theorem for integrals is satisfied.

   Solution: The average value is $\frac{1}{\sqrt{5}-1}\int_1^{\sqrt{5}} 2x(x^2 - 1)^{3/2}\ dx = \frac{1}{\sqrt{5}-1}\frac{2}{5}(x^2 - 1)^{5/2}|_1^{\sqrt{5}} = \frac{1}{\sqrt{5}-1}\frac{64}{5} \approx 10.36$. To find c solve the equation $2x(x^2 - 1)^{3/2} = 10.36$. The graphing calculator solver feature gives $c \approx 1.75$ for this.

3. (15 points) The elastic force of a spring satisfies Hooke's Law $F(x) = kx$ with spring stiffness k. To extend the spring from its equilibrium $x = 0$ to $x = 0.20$ meters, a force of 8 Newtons is required. Find k. Then find the amount of work W done extended the spring from equilibrium to $x = 0.80$ meters.

   Solution: $.2k = 8 \Rightarrow k = 40$. Then $W = \int_0^{.8} 40x = 20x^2|_0^{.8} = 0.64(20) = 12.8J$.

4. (15 points) Use the numerical integration feature on your calculator to find the length of the curve $y = \sin x$ between 0 and π.

   Solution: Using the numerical integration feature, we find $\int_0^{\pi} \sqrt{1 + (\cos x)^2}\ dx \approx 3.820$.

5. (50 points) In the first quadrant $x \geq 0, y \geq 0$, a region R is bounded above by the curve $y = 2x - x^2$ and bounded below by $y = x^3$.

   (a) Set up an integral whose value is the volume of the solid obtained by rotating R around the x-axis.

   Solution: $\pi \int_0^1 (2x - x^2)^2 - (x^3)^2 \; dx$

   (b) Set up an integral whose value is the volume of the solid obtained by rotating R around the y-axis.

   Solution: Slicing, $\pi \int_0^1 (y^{1/3})^2 - \left(1 - \sqrt{1-y}\right)^2 \; dy$.

   Using shells, $2\pi \int_0^1 x(2x - x^2 - x^3) \; dx$.

   (c) Set up an integral whose value is the volume of the solid obtained by rotating R around the line $y = -1$.

   Solution: $\pi \int_0^1 (2x - x^2 + 1)^2 - (x^3 + 1)^2 \; dx$

   (d) Using the cylindrical shells technique, set up an integral whose value is the volume of the solid obtained by rotating R around the line $x = -1$.

   Solution: $2\pi \int_0^1 (x+1)(2x - x^2 - x^3) \; dx$

   (e) Using the slicing technique, set up an integral whose value is the volume of the solid obtained by rotating R around the line $x = -1$.

   Solution: Using slicing, $\pi \int_0^1 (1 + y^{1/3})^2 - (2 - \sqrt{1-y})^2 \; dy$.

6. (15 points) Calculate the moments M_x and M_y and the center of mass of the lamina R of density $\rho = 1$ and shape given by $R = \{(x, y) \mid 0 \leq x \leq 2 \text{ and } 0 \leq y \leq \sqrt{4 - x^2}\}$.

   Solution: The region R is the quarter of the circle $x^2 + y^2 = 4$ that lies in the first quadrant. Thus $M_x = \int_0^2 \frac{1}{2}(f(x))^2 \; dx = \frac{1}{2}\int_0^2 (\sqrt{4 - x^2})^2 \; dx = \frac{1}{2}\int_0^2 4 - x^2 \; dx = \frac{8}{3}$. The mass m is the product of the density $\rho = 1$ and the area, which is clearly $\pi 2^2/4 = \pi$. Because of the symmetry of the lamina about the line $y = x$, $M_y = M_x$. So the center of mass is the point $(x, y) = (\frac{8}{3\pi}, \frac{8}{3\pi})$.