March 21, 2005 **Name** ______________________

On all the following questions, **show your work.** There are 121 points available on this test. Do not try to do all the problems. Try to find four or five that you you can do well.

1. (10 points) Let R denote the "triangular" region in the plane bounded by the lines $y = -x$, $y = 4$ and the curve $y = x^2, x \geq 0$. Set up an integral $\int_a^b f(y)dy$ whose value is the area of the region R. What is the area of R?

   Solution: $\int_a^b f(y)dy = \int_0^4 \sqrt{y} + y dy = 2y^{3/2}/3 + y^2/2|_0^4 = 16/3 + 16/2 = 40/3$.

2. (10 points) To find the length of the curve defined by $x = f(t) = t^2 - \sin t$ and $y = g(t) = t^3 + \cos t$ from the point $(0, 1)$ to the point $(\pi^2, \pi^3 - 1)$, you'd have to compute

$$\int_a^b k(t)dt.$$

   (a) What is the value of a?
   Solution: $t = 0$ gives $(x, y) = (f(0), g(0)) = (0, 1)$.

   (b) What is the value of b?
   Solution: $t = \pi$ gives $(x, y) = (f(\pi), g(\pi)) = (\pi^2, \pi^3 - 1)$.

   (c) What is $k(t)$?
   Solution: $k(t) = \sqrt{(2t - \cos t)^2 + (3t^2 - \sin t)^2}$.

3. (65 points) Let R be the region bounded by the graphs of $y = \sqrt{x}$, the line $x = 4$ and the x-axis.

   (a) What is the area of R?

   Solution: $A(R) = \int_0^4 \sqrt{x}\ dx = \frac{2}{3}x^{3/2}|_0^4 = 16/3.$

   (b) Find the volume generated by revolving R about the x-axis using the slicing method.

   Solution: $V = \int_0^4 \pi(\sqrt{x})^2\ dx = \pi \int_0^4 x\ dx = 8\pi.$

   (c) Find the volume generated by revolving R about the x-axis using the shelling method.

   Solution: $V = \int_0^2 2\pi y(4 - y^2)\ dy = 2\pi \int_0^2 4y - y^3 dy = 8\pi.$

(d) Find the volume generated by revolving R about the y-axis using the slicing method.
Solution: We get washers: $V = \int_0^2 \pi(4^2 - (y^2)^2)\, dy = \pi(16y - y^5/5|_0^2) = 128\pi/5$.

(e) Find the volume generated by revolving R about the y-axis using the shelling method.
Solution: $V = 2\int_0^4 \pi x(\sqrt{x})\, dx = (2\pi)2x^{5/2}/5|_0^4) = 128\pi/5 \approx 80.42$.

(f) Set up an integral whose value is the volume of the solid generated by revolving R about the line $y = -5$.
Solution: We get washers: $V = \pi\int_0^4 (5 + \sqrt{x})^2 - 5^2\, dx$.

(g) Set up an integral whose value is the volume of the solid generated by revolving R about the line $x = 8$.
Solution: Using shelling, $V = 2\pi\int_0^4 (8 - x)\sqrt{x}\, dx$. On the other hand, we could use slicing to get $V = \pi\int_0^2 (8 - y^2)^2 - 4^2\, dy$.

(h) Set up an integral whose value is the length of the curve

$$C = \{(x, \sqrt{x}) \mid 0 \leq x \leq 4\}.$$

Solution: $L = \int_0^4 \sqrt{1 + \dfrac{1}{4x}}\, dx \approx 4.65$.

4. (12 points) A man drives his car down a road in such a way that its velocity (in m/s) at time t (seconds) is

$$v(t) = 3t^{1/2} + 1.$$

Find the car's average velocity (in m/s) between $t = 1$ and $t = 4$.

Solution: $V_{ave} = \frac{1}{4-1}\int_1^4 3t^{1/2} + 1\ dt = \frac{1}{3}(2t^{3/2} + t|_1^4) = \frac{1}{3}(2 \cdot 8 + 4 - (2+1)) = \frac{17}{3}$ Note that the distance travelled by the car is simply the integral $\int_1^4 3t^{1/2} + 1\ dt$.

5. (12 points) Find the mean value of the function $f(x) = |7 - x|$ on the closed interval $[5, 11]$. Then find the value c guaranteed by the Mean Value Theorem for Integrals. Ie, find c such that $f(c) = f_{ave}$.

Solution: This problem is easy to do geometrically. Sketch the graph of $f(x) = |7-x|$ to see that the region bounded above by the function consists of two triangles with areas 2 and 8. So $\int_5^{11} |7 - x|\ dx$ is 10. This means the average value is $10/6 = 5/3$. To find c, solve the equation $|7 - x| = 5/3$. This yields the two values $c = 16/3$ and $c = 26/3$. This problem came from webwork.

6. (12 points) Find the centroid $(\bar{x}, \bar{y})$ of the region bounded by:

$$y = 9x^2 + 9x, \quad y = 0, \quad x = 0, \quad \text{and} \quad x = 1.$$

Solution: The area is given by $A = \int_0^1 9(x^2 + x)\ dx = 15/2$. The moments M_y and M_x are given by $M_y = \int_0^1 9x(x^2 + x)\ dx = 5.25$ and $M_x = \int_0^1 \frac{1}{2}9^2(x^2 + x)^2\ dx \approx 41.81$. Therefore $(\overline{x}, \overline{y}) \approx (5.25/7.5, 41.85/7.5) = (0.70, 5.58)$.