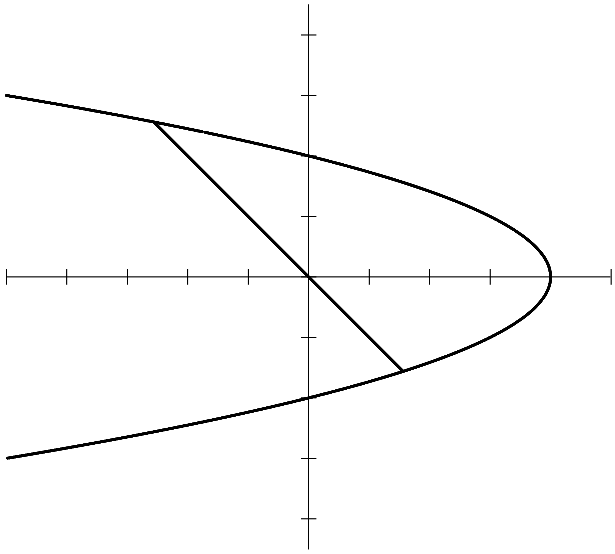
October 25, 2005 **Name** ____________________________

On all the following questions, **show your work.** There are 137 points available on this test. Do not try to do all the problems. Try to find four or five that you you can do well.

1. (12 points) Let R denote the region in the plane enclosed by $x + y^2 = 42$ and $x + y = 0$.

   (a) Sketch the region R.

   Solution:



   (b) Then find the area of the region. To do this, set up an integral $\int_a^b f(y)dy$ whose value is the area of the region R. What is the area of R?

   Solution: Solve the equation $x = -y = 42 - y^2$ to get $y = -6$ and $y = 7$. The area is $A = \int_{-6}^{7} X_r - X_l dy = \int_{-6}^{7} 42 - y^2 + y\, dy = 42y + y^3/3 + y^2/2|_{-6}^{7} \approx$ 366.16.

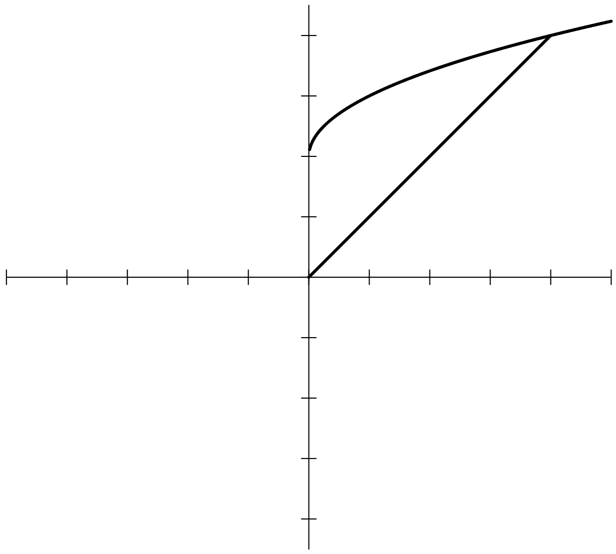
2. (15 points) Find the mean value of the function $f(x) = \sqrt{x-4}$ on the closed interval $[4, 13]$. Then find the value c guaranteed by the Mean Value Theorem for Integrals. Ie, find c such that $f(c) = f_{ave}$.

   Solution: First note that $\int_4^{13} \sqrt{x-4}\, dx$ is 18. This means the average value is $18/9 = 2$. To find c, solve the equation $f(c) = \sqrt{c-4} = 2$. This yields the value $c = 8$.

3. (75 points) Let R be the region bounded by the graph of $y = 2 + \sqrt{x}$, the y-axis, the line $y = x$.

   (a) Sketch R.

   Solution:



   (b) What is the area of R?

   Solution: $A(R) = \int_0^4 2+\sqrt{x} - x \, dx = 2x + \frac{2}{3}x^{3/2} - x^2/2|_0^4 = 8 + 16/3 - 8 = 16/3$.

   (c) Find the volume generated by revolving R about the x-axis.

   Solution: $V = \int_a^b \pi(R^2 - r^2)\, dx \int_0^4 \pi(2+\sqrt{x})^2 - x^2 \, dx = \pi \int_0^4 4 + 4\sqrt{x} + x - x^2 \, dx = \pi \left(4x + 4 \cdot 2x^{3/2}/3 + x^2/2 - x^3/3|_0^4\right) = 24\pi$.

   (d) Find the volume generated by revolving R about the y-axis using the shelling method.

   Solution: $V = \int_0^4 2\pi x \left[(2 + \sqrt{x}) - x\right] \, dx = 2\pi \int_0^4 2x + x^{3/2} - x^2 \, dx = 2\pi[^2 x + 2x^{5/2}/5 - x^3/3|_0^4 = 224\pi/15$.

   (e) Find the volume generated by revolving R about the line $y = -1$.

   Solution: We get washers: $V = \int_0^4 \pi(R^2 - r^2) \, dx = \pi \int_0^4 ((3 + \sqrt{x})^2 - (x+1)^2 \, dx = \pi \int_0^4 9 + 6\sqrt{x} - x - x^2 - 1 \, dx = 104\pi/3$.

(f) Set up an integral whose value is the volume of the solid generated by revolving R about the line $x = 5$.

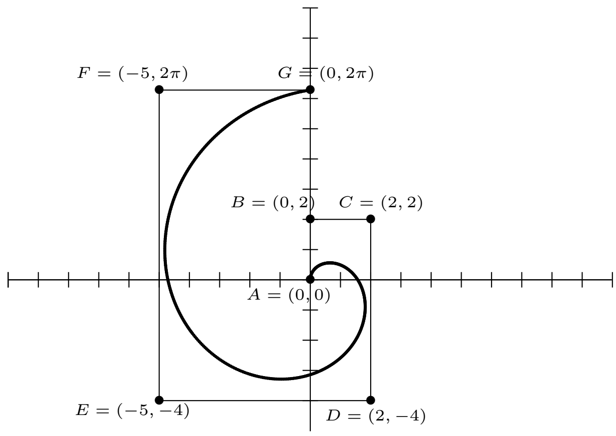
Solution: Using shells, $V = 2\pi \int_0^4 (5 - x)[2 + \sqrt{x} - x]\ dx$.

(g) Set up an integral whose value is the length of the curve

$$C = \{(x, 2 + \sqrt{x}) \mid 0 \le x \le 4\}.$$

Solution: $L = \int_0^4 \sqrt{1 + \frac{1}{4x}}\ dx$. Incidentally, this value is approximately 4.65.

4. (20 points) Consider the curve defined by $x = f(t) = t \sin t$ and $y = g(t) = t \cos t$, $0 \le t \le 2\pi$.



(a) (2) We can get a lower bound for the length of the curve by measuring the straight line distance from A to G. Within 0.01 what is the distance between these two points?

Solution: 2π.

(b) (4) We can get an upper bound for the length of the curve by measuring the polygonal distance from A to G using B, C, D, and E as vertices of the polygon. In other words, find the sum of the distances $AB + BC + CD + DE + EF + FG$.

Solution: $26 + 2\pi \approx 32.28$.

(c) (14) Find the exact length of the curve. If you get an integrand you cannot antidifferentiate, use the numerical integration application on your calculator. If you use such, write clearly your integrand.

Solution: The derivatives are $f'(t) = \sin t + t\cos t$ and $g'(t) = \cos t - t\sin t$. Squaring and adding produces $\sqrt{f'(t)^2 + g'(t)^2}$
$= \sqrt{\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + \cos^2 t - 2t\sin t\cos t + t^2\sin^2 t}$
$= \sqrt{1+t^2}$. The length is therefore $\int_0^{2\pi} \sqrt{1+t^2}\ dt$. Using the numerical integration feature on the calculator, we get $L \approx 21.26$

5. (15 points) A tank in the shape of an inverted right circular cone has height 8 meters and radius 14 meters. It is filled with 6 meters of hot chocolate. Find the work required to empty the tank by pumping the hot chocolate over the top of the tank. Note: the density of hot chocolate is $\delta = 1520kg/m^3$.

   Solution: $W = \int_0^6 Vol\cdot density\cdot gravity\cdot distance\ dx = \rho\cdot 9.8\pi \int_0^6 (14x/8)^2(8-x)\ dx = 1520\cdot 49\cdot 9.8\cdot \pi/16 \int_0^6 8x^2 - x^3\ dx \approx 11495988\pi J$. or Set up the coordinates so that the top of the cone is at 0 and the bottom at 8. The $r = 14(8-x)/8$ and $W = \int_2^8 Vol\cdot density\cdot gravity\cdot distance\ dx = \rho\cdot 9.8\pi\int_2^8 x(7/4)^2(8-x)^2\ dx = (1520\cdot 49\cdot 9.8\cdot \pi/16)\int_2^8 x(64-16x+x^2)dx \approx 36115711.44J$.