April 4, 2006 **Name** ______________________________

On all the following questions, **show your work.** There are 150 points available on this test. Do not try to do all the problems. Try to find four or five that you you can do well. Except in problems 1, 4b, and 4c, there is no penalty for using the numerical integrator (fnInt on the TI's). However, you must display the integral explicitly to get credit.

1. (10 points) Find the area of the region bounded by $f(x) = x^2 + 6$, $g(x) = x$, $x = -2$, and $x = 5$.

   Solution: $A(R) = \int_{-2}^{5} x^2 + 6 - x \; dx = x^3/3 + 6x - x^2/2|_{-2}^{5} = 75\frac{5}{6}$.

2. (10 points) Sketch the region enclosed by the curves $y = 4x^2, y = x^2 + 3$. You want to find the area of the region. Decide whether to integrate with respect to x or y. Do not integrate. Just set up the integral.

   Solution: $A(R) = \int_{a}^{b} top - bottom \; dx = \int_{-1}^{1} x^2 + 3 - 4x^2 \; dx.$

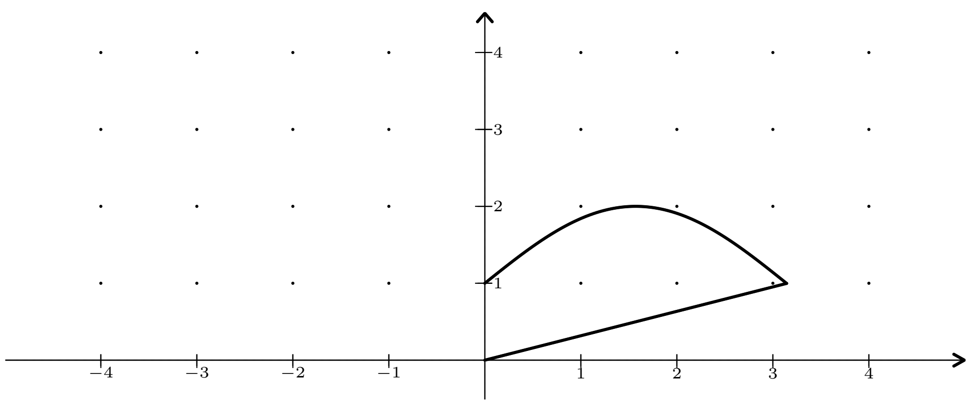
3. (10 points) Sketch the region enclosed by $x + y^2 = 2$ and $x + y = 0$. You want to find the area of the region. Decide whether to integrate with respect to x or y. Do not integrate. Just set up the integral.

   Solution: $A(R) = \int_{c}^{d} x_R - x_L \; dy = \int_{-1}^{2} (2-y^2) - (-y) \; dy = \int_{-1}^{2} 2 + y - y^2 \; dy.$

4. (70 points) Let R be the region bounded by the graph of $y = 1 + \sin x$, the y-axis, and the line $y = x/\pi$.

   (a) Sketch R.

   Solution:



(b) What is the area of R? (2 points for correct numerical value only)

Solution: $A(R) = \int_0^\pi 1 + \sin x \ - x/\pi \ dx = x - \cos x - x^2/(2\pi)|_0^\pi = \pi - \cos\pi - \pi^2/(2\pi) - (-\cos 0) = \pi/2 + 2 \approx 3.571$.

(c) Find the volume generated by revolving R about the x-axis. (2 points for correct numerical value only) You might need to know the following identity: $\sin^2 x = \dfrac{1-\cos 2x}{2}$.

Solution: $V = \int_a^b \pi(R^2 - r^2)\ dx = \pi\int_0^\pi (1+\sin x)^2 - (x/\pi)^2\ dx = \pi\left(x - 2\cos x + x/2 - \dfrac{\sin 2x}{4} - \dfrac{x^3}{3\pi^2}\right)|_0^\pi = \pi\left[\pi + 2 + \pi/2 - \pi/3 - (-2)\right] = \pi\left[\frac{7\pi}{6} + 4\right] \approx 24.081$.

(d) Find the volume generated by revolving R about the y-axis using the shelling method.

Solution: $V = \int_0^\pi 2\pi x\left[(1+\sin x) - x/\pi\right]\ dx = 2\pi\int_0^\pi x + x\sin x - x^2/\pi\ dx = 2\pi[x^2/2 - x\cos x + \sin x - x^3/(3\pi)|_0^\pi \approx 2\pi\cdot 4.7865 \approx 30.075$.

(e) Find the volume generated by revolving R about the line $y = -1$.

Solution: We get washers: $V = \int_0^\pi \pi(R^2 - r^2)\ dx = \pi\int_0^\pi ((2+\sin x)^2 - (1 + x/\pi)^2\ dx \approx 14.806\pi \approx 46.516$.

(f) Set up an integral whose value is the volume of the solid generated by revolving R about the line $x = 4$. Do not integrate.

Solution: Using shells, $V = 2\pi\int_0^\pi (4-x)[1 + \sin x - x/\pi]\ dx$.

(g) Set up an integral whose value is the volume of the solid generated by revolving R about the line $y = 2$. Do not integrate.

Solution: Slicing, $V = \pi\int_0^\pi (2 - x/\pi)^2 - (2-(1+\sin x))^2\ dx = \pi\int_0^\pi (2 - x/\pi)^2 - (1-\sin x)^2\ dx$.

5. (10 points) Find the length of the curve defined by

$$y = 2x^3 + 2x$$

from the point (-2,-20) to the point (3,60).

Solution: First compute $y' = 6x^2 + 2$ so $y'^2 = 36x^4 + 24x^2 + 4$, so $L(C) = \int_{-2}^{3} \sqrt{36x^4 + 24x^2 + 4 + 1}\ dx \approx 80.3756$

6. (10 points) Find the mean value of the function $f(x) = |5 - x|$ on the closed interval $[4, 8]$.

Solution: The region bounded by f is the union of two triangles. By the definition, $f_{ave} = \dfrac{1}{8-4}\displaystyle\int_4^8 |5 - x|\ dx = \dfrac{1}{4}\left[\dfrac{1}{2} + \dfrac{9}{2}\right] = \dfrac{5}{4}$, since f is a nonnegative function (hence $\int f =$ area of the region bounded by f).

7. (10 points) A force of 1 pound is required to hold a spring stretched 0.1 feet beyond its natural length of six feet. How much work (in foot-pounds) is done in stretching the spring from six feet to seven feet?

Solution: Find k by solving $f(x) = kx$, ie $f(0.1) = k(0.1) = 1$lb to get $k = 10$. Then integrate to find the work done. $W = \int_0^1 10x\ dx = \dfrac{10x^2}{2}\Big|_0^1 = 5$ft-lbs. of work.

8. (10 points) The base of a certain solid is the region bounded above by the graph of $y = f(x) = 25$ and below by the graph of $y = g(x) = 9x^2$. Cross-sections perpendicular to the y-axis are squares. Use the formula

$$V = \int_a^b A(y)\, dy$$

to find the volume of the solid.

Solution: Since the cross-sections are squares, the area function is given by $A(y) = \left(2\sqrt{\frac{y}{9}}\right)^2 = 4y/9$. So the volume is $\int_0^{25} 4y/9\ dy = 2y^2/9|_0^{25} = 1250/9 \approx$ 137.7

9. (10 points) The masses m_i are located as given. Find the center of mass of the system. Mass $m_1 = 4$ is located at $(7, -4)$, $m_2 = 2$ at $(9, -5)$ and $m_3 = 6$ at $(3, -4)$.

Solution: $\overline{x} = \dfrac{M_y}{M} = \dfrac{4 \cdot 7 + 2 \cdot 9 + 6 \cdot 3}{4 + 2 + 6} = \dfrac{64}{12} = 5.\overline{3}$ and $\overline{y} = \dfrac{M_x}{M} = \dfrac{4(-4) + 2(-5) + 6(-4)}{12} = \dfrac{-16 - 10 - 24}{12} = -25/6$. So $(\overline{x}, \overline{y}) = (16/3, -25/6)$.