April 27, 2006 **Name** ____________________________

On all the following questions, **show your work.** There are 130 points available on this test. Do not try to do all the problems.

1. (20 points) Determine whether each of the **sequences** below converges. If so, find the limit.

   (a) $\lim_{n\to\infty} \frac{11(2^n)+10}{8(2^n)}$
   Solution: The sequence is the sum of two sequences, one of which is the constant 11/8 and the other a sequence with limit 0, so the sequence converges to 11/8.

   (b) $\lim_{n\to\infty} \frac{35}{3^n} + 2\arctan(n^2)$
   Solution: Again, the sequence is the sum of two sequences, one of which is has limit 0. The other $2\arctan(n^2)$ converges to π.

   (c) $\lim\limits_{n\to\infty} n(0.7)^n$
   Solution: This sequence has limit 0 by H'lospital's rule.

   (d) $\lim\limits_{n\to\infty} \dfrac{(2n+1)^2}{(3n-1)^2}$

   Solution: Rewrite the problem as $\lim\limits_{n\to\infty} \dfrac{4n^2+4n+1}{9n^2-6n+1}$ which is 4/9 by the usual method of dividing both numerator and denominator by n^2.

2. (25 points) Label each of the following series with C or D, where C stands for Convergent, D stands for Divergent. State the reason for your answer.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+3}$

   Solution: This series diverges by the limit comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

   Solution: This series converges by the limit comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}^2$

   (c) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$
   Solution: This series converges by the ratio test.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{7+5^n}{10^n}$

Solution: This series converges because it is the sum of two geometric series both with ratios less than 1.

(e) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

Solution: This series diverges by the integral test.

3. (30 points) Test each of the following series for convergence. name the test you use. Two points for right convergence answer, three points for correct test. If a series converges conditionally, but not absolutely, say so.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+2)!}{(n!)^2}$

   Solution: Use the ratio test: $|a_{n+1}/a_n| = \frac{(2n+4)!n!n!}{(n+1)!(n+1)!(2n+2)!} = \frac{(2n+4)(2n+3)}{(n+1)(n+1)} \to 4$. Thus, the limit is not less than 1, so the series diverges.

   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(5+n)4^n}{(n^2)3^{2n}}$

   Solution: Again, use the ratio test, this time getting a limiting ratio of 4/9, so the series converges.

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

   Solution: This is a p-series with $p = 3/2$ so the series converges. Alternatively, use the integral test.

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n\pi}$

   Solution: Compare this with the alternating harmonic series. It converges by the alternating series test.

   (e) $\displaystyle\sum_{n=1}^{\infty} (-1)^n n^{-1} \ln(n+3)$

   Solution: This converges by the alternating series test. The terms go to zero and the series of absolute values is monotonic.

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{5+\sin(n)}{\sqrt{n}}$

Solution: Limit compare or compare this with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. It diverges.

4. (10 points) Find the value of

$$\int_2^{\infty} \frac{dx}{(9x-2)^8}$$

Determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(9n-2)^8}$ is convergent or divergent.

Solution: Use substitution to rewrite the problem as $\frac{1}{9}\int_{16}^{\infty} u^{-8}\, du$, which is easily seen to converge to $\frac{1}{63 \cdot 16^7}$. So the series converges by the integral test.

5. (15 points) Find the interval of convergence for the given power series.

$$\sum_{n=1}^{\infty} \frac{(x-6)^n}{n(-6)^n}$$

In particular, be sure to discuss convergence at the endpoints of the interval.

Solution: The ratio test leads to $|\frac{(x-6)^{n+1}}{(x-6)^n} \frac{n}{(n+1)} \frac{(-6)^n}{(-6)^{n+1}}| \to |-(x-6)/6|$, which is less than 1 when $|x-6| < 6$. So the interval of convergence is $(0, 12]$ since at $x = 12$, the series alternates and at $x = 0$, it does not.

6. (15 points) Suppose that $\dfrac{5x}{(6+x)} = \displaystyle\sum_{n=0}^{\infty} c_n x^n$.

   Find the first few coefficients.

   Solution: Rewrite the function as $\frac{5x}{6} \cdot \frac{1}{(1-(-\frac{x}{6})}$ and then use the basic idea to get $\dfrac{5x}{(6+x)} = 5\displaystyle\sum_{n=0}^{\infty}(-x/6)^{n+1}$. Equating this with $\sum_{n=0}^{\infty} \frac{f^{(n)}}{n!} x^n$ leads to the coefficients.

   $c_0 =$ ___________

   $c_1 =$ ___________

   $c_2 =$ ___________

   $c_3 =$ ___________

   $c_4 =$ ___________

   Solution: $c_0 = 0$ $c_1 = -5/36$ $c_2 = 5/216$ $c_3 = -5/1296$ $c_4 = 5/6^5$

   Find the radius of convergence R of the power series.

   $R =$ ___________ .

   Solution: $R = 6$

7. (15 points) Consider the series $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ obtained from the harmonic series by replacing every third plus sign with a minus sign. Very few of the theorems in class apply to this series. Its not alternating. Does it converge or diverge? Discuss your reasoning. If you think it converges, does it converge conditionally or absolutely?

   Solution: The series diverges. Group the terms in 3s and compute. Note that $1+1/2-1/3 > 1, 1/4+1/5-1/6 > 1/4$, etc. In general $a_{3n+1}+a_{3n+1}+a_{3n+2} = 1/(3n+1) + 1/(3n+2) - 1/(3n+3) > 1/(3n+1)$. It follows that our series diverges, comparing it with $\frac{1}{4}\sum \frac{1}{n}$.