Math 3166 Combinatorics Final H. Reiter May 7, 1993

Name ________________________

If you want your grade posted, record the last four digits of your student ID number in the blank ______

The first seven problems are required. You may choose any five others. Show all work.

1. Find the number of possible secret codes in six color, four position Mastermind, following the response ! ! + to the initial guess ABCD.

2. Four letters are randomly selected from M A T H E M A T I C S. What is the probability that you can spell MATH with the four letters?

3. Find the number of solutions of the integer system

$$u + v + w \leq 8,\ u \geq 0,\ v \geq 0,\ \text{and } w \geq 0$$

4. For how many positive integers n less than or equal to 1000 is it true that n is a multiple of 2, 3, or 7?

5. Consider the polynomial $G(x, y) = (2x - y - 1)^6$. The terms are all of the form $a_{ij}x^iy^j$.

   (a) How many terms are there?

   (b) What is the sum of all the coefficients? I. E., what is $\sum a_{ij}$?

   (c) List all the terms of the polynomial which have the form $a_{3j}x^3y^j$?

6. Find the general solutions to the recurrence relation $a_n = 2a_{n-1} - a_{n-2}, n \geq 3$.

   (a) Find a specific solution in case $a_1 = 1$ and $a_2 = 1$.

   (b) Find a specific solution in case $a_1 = 1$ and $a_2 = 2$.

7. The sequence of Fibonacci $\{f_n\}$ numbers is defined by the recurrence $f_n = f_{n-1} + f_{n-2}$ together with the initial values $f_o = 1$ and $f_1 = 1$. Prove that for all $n > 0$,

$$f_0^2 + f_1^2 + \ldots + f_n^2 = f_n f_{n+1}.$$

8. Six identical dice are rolled. How many different outcomes are possible? What is the probability that the sum of all six dice

   (a) is even?

   (b) is a multiple of three?

   (c) is a multiple of four?

9. How many functions are there from a nine element set into a two element set? How many of these functions are 'onto'. Answer the same two questions when the two element set is changed to a three element set.

10. Give both a committee assignment proof and a block walking proof of

$$\binom{n}{k} = \binom{n}{n-k}.$$

11. Find a generating function for the number of ways to make r cents out of an unlimited supply of pennies, nickels, dimes and quarters.

12. How many <u>even</u> integers greater than 75,000 have the following properties:
(1) the digits are distinct and
(2) the digits 3 and 4 do not occur in the number.

13. How many <u>even</u> integers less than 75,000 have the following properties:
(1) the digits are distinct and
(2) the digits 3 and 4 do not occur in the number.

14. Let S_n denote the number of non-empty subsets of the set $\{1, 2, 3, \ldots, n\}$ which do not contain any two consecutive integers. For example, $S_1 = 1, S_2 = 2$, and $S_3 = 4$(the subsets are $\{1\}\{2\}\{3\}\{1, 3\}$).
Find a recurrence relation for S_n.

15. Two numbers a and b are randomly selected without replacement from the set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. What is the probability that $ab + a + b + 1$ is even? Suppose three numbers a, b, and c are randomly selected from the same set without replacement. What is the probability that $abc+ab+bc+ac+a+b+c+1$ is even?

16. Suppose a chess king can move from a square to any square that it touches, even if it touches only at a corner. How many paths of length 14 are there from the square marked S to the one marked F in the 3 by 15 grid? Let a_n denote the number of paths of length $n - 1$ from the S to the F in a 3 by n grid. Find a recurrence relation for a_n. For example, $a_2 = 1$ and $a_3 = 3$.

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