Math 3166   Combinatorics Final   H. Reiter   Spring 1989

Name _____

If you want your grade posted, record the last four digits of your student ID number in the blank _____

The first seven problems are required. You may choose any five others. Show all work.

1. Use the binomial theorem to write $(1 + 2x)^5$ as a polynomial in x.

2. Expand $(1 + x + y)^5$.

3. A box contains marbles of three colors: red, white, and blue. The number of blue marbles is at least half the number of white marbles and at most one third the number of red marbles. The number of marbles which are white or blue is a least 55. Find the minimum number of red marbles.

4. Find the generating function for the number of ways to make r cents in change in pennies, nickels, and dimes.

5. Find the coefficient of x^{10} in $x^2(1 - x)^{-10}$.

6. How many sequences of length 10 made from the letters a, b, c and d are there for which no pair of consecutive letters are identical?

7. A $3 \times 3 \times 3$ cube is build from 27 unit cubes. Imagine the cube is located so that one vertex is at $(0, 0, 0)$ and another at $(3, 3, 3)$. How many paths of length 9 are there along edges of the component unit cubes from $(0, 0, 0)$ to $(3, 3, 3)$?

8. In the diagram below, how many paths along the line segments drawn are there from vertex marked S to the one marked F which use either 5 or 6 segments.
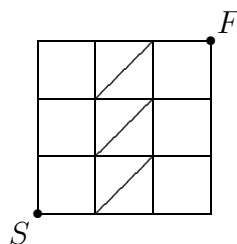
9. Prove that for all integers $n > 0$, $2^{2n} - 1$ is divisible by 3.

10. Five people are sitting at a round table. Let f and m be the number of people sitting next to at least one female and at least one male, respectively. Find the number of possible ordered pairs (f, m).

11. In a certain cross-country meet between two teams of five runners each, a runner who finishes in the nth position contributes n to his team's score. The team with the lower score wins. If there are no ties among the runners, how many different winning scores are possible?

12. Let n be a positive integer. If the equation $2x + 2y + z = n$ has 28 integral solutions, what are the possible vaules of n?

13. Find the number of functions from a 10 element set into a 3 element set. Then find the number of such functions which map the 10 elelment set *onto* the 3 element set.

14. Find a set S of six positive integers $a_1 < a_2 < a_3 < a_4 < a_5 < a_6$ so that the set of distances among the pairs of elements of S has the cardinality $\binom{6}{2} = 15$. What is the smallest possible value for a_6?

15. Compute the rook polynomial of the 5×5 board with both diagonals shaded, and use this to find the number of permutations π of $\{1, 2, 3, 4, 5\}$ such that $\pi(i) \notin \{i, 6 - i\}$.

16. Find the rook polynomial of the standard 8×8 board B. Recall that the dark and light squares alternate. Use this polynomial $R(x, B)$ to count the number of ways to place eight independent rooks on the light squares of B.

17. Provide a committee assignment proof that $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.

18. Provide a block walking proof that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

19. At the end of a professional bowling tournament, the top five bowlers have a playoff. First, #5 bowls against #4. The loser gets fifth place, and the winner bowls #3. This goes on until the winner of the third match bowls #1 for the top spot. How many different orders of finish are there for bowlers #1 through #5?

20. Given 20 points on a circle, how many convex quadrilaterals can be inscribed in the circle with all the vertices among the 20 points.