

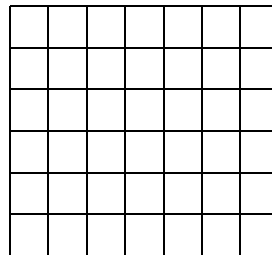
March 5, 1999

Your name _____

Show your work!

1. (20 points) Let $S = \{1, 2, 3, 4, 5, 6, 7\}$.
 - (a) How many non-empty subsets does S have?
 - (b) How many subsets of S have no odd numbers as members?
 - (c) How many subsets of S have exactly 4 elements?
 - (d) How many four element subsets of S contain exactly two odd numbers?
 - (e) How many four digit numbers can be made using the digits of S if a digit may be used repeatedly?
 - (f) How many four digit numbers can be made using the digits of S if a digit may be used only once?
 - (g) How many even four digit numbers can be made using the digits of S if a digit may be used only once?
2. (15 points) Of 28 students taking at least one subject, the number taking Mathematics and English only equals the number taking Mathematics only. No student takes English only or History only, and six students take Mathematics and History, but not English. The number taking English and History only is twice the number taking just Math. The number taking History is 18. If the number taking all three subjects is even and non-zero, what is the number taking Mathematics only?
3. (15 points) Look at the four equations below.
$$\begin{array}{rcl} 2 & = & 2 \cdot 1 \\ 2 + 4 & = & 3 \cdot 2 \\ 2 + 4 + 6 & = & 4 \cdot 3 \\ 2 + 4 + 6 + 8 & = & 5 \cdot 4 \end{array}$$
 - (a) Write the next three equations in the sequence.
 - (b) If the four equations above correspond to $k = 1, 2, 3$, and 4, what is the n^{th} equation?
 - (c) Prove by mathematical induction that the n^{th} equation is true for all integers $n \geq 1$.
4. (15 points) There are 33 students in the class and sum of their ages 430 years. Is it true that one can find 20 students in the class such that the sum of their ages is greater than 260? Explain in detail.

5. (15 points) Consider the arrangement of the letters *ANNIVERSARY*.
- How many such arrangements are there?
 - When these arrangements are put in alphabetical order, what is the arrangement in position 50?
 - In alphabetical order, the arrangement *AAEINNRRSVY* comes first. What is the position of the arrangement *EAAINNRRVYS*?
6. (15 points) An experiment consists of rolling a balanced die three times, thus obtaining an ordered triple of integers (x, y, z) . Calculate the following probabilities.
- $P(x + y + z \leq 10.5)$.
 - $P(x \cdot y \cdot z \text{ is even})$.
 - $P(x + y = z)$.
7. (15 points) Tilings.
- How many ways can a 3×3 board be tiled by four dominos ($\square\square$) and one monomino (\square)?
 - There are **four** tilings of a 3×4 board using L-shaped triominoes ($\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$). How many tilings of a 3×7 board are there with six L-shaped triominoes and one regular triomino ($\square\square\square$)?
 - How many tilings of a 4×4 board are there with five L-shaped triominoes and one monomino (\square)?
8. (15 points) Six fair identical cubical dice are rolled.
- How many outcomes are possible?
 - What is the probability that three pairs are obtained? That is, what is the probability that the multiset of outcomes is u, u, v, v, w, w where u, v , and w are distinct?
 - What is the probability that six different numbers result?



9. (20 points) Consider the 6×7 grid of unit squares.

- (a) How many rectangles are the union of one or more of the unit squares of the grid? (A 1×1 square counts as a rectangle.)
- (b) How many of the rectangles in (a) are squares?
- (c) How many of the rectangles have area 12?
- (d) How many of the rectangles are the union of an odd number unit squares?
- (e) How many of the rectangles have a perimeter that is a multiple of 3?

SOLUTIONS

1. (20 points) Let $S = \{1, 2, 3, 4, 5, 6, 7\}$.

- (a) How many non-empty subsets does S have?

Solution. $2^7 - 1 = 127$. Alternatively, $\binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \cdots + \binom{7}{7} = 127$.

- (b) How many subsets of S have no odd numbers as members?

Solution. $2^3 = 8$

- (c) How many subsets of S have exactly 4 elements?

Solution. $\binom{7}{4} = 35$.

- (d) How many four element subsets of S contain exactly two odd numbers?

Solution. $\binom{4}{2} \cdot \binom{3}{2} = 18$.

- (e) How many four digit numbers can be made using the digits of S if a digit may be used repeatedly?

Solution. $7^4 = 2401$.

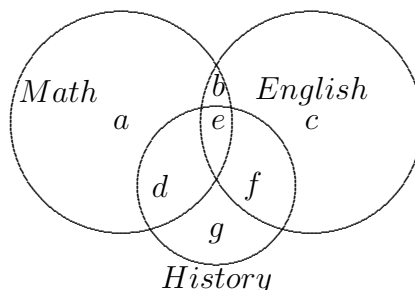
- (f) How many four digit numbers can be made using the digits of S if a digit may be used only once?

Solution. $P_4^7 = 7 \cdot 6 \cdot 5 \cdot 4 = 840$.

- (g) How many even four digit numbers can be made using the digits of S if a digit may be used only once?

Solution. $4 \cdot 5 \cdot 6 \cdot 3 = 360$.

2. (15 points) Of 28 students taking at least one subject, the number taking Mathematics and English only equals the number taking Mathematics only. No student takes English only or History only, and six students take Mathematics and History, but not English. The number taking English and History only is twice the number taking just Math. The number taking History is 18. If the number taking all three subjects is even and non-zero, what is the number taking Mathematics only?



Solution. Let a, b, c, d, e, f, g denote the number of students in each of the regions above. Then each of the equations $a + b + c + d + e + f + g = 28$, $a = b$, $c = g = 0$, $f = 2a$, and $d = 6$ holds. It follows that $e + f = 12$ and $a + b + e + f = 22$ from which it follows that $a + b = 10$ and $a = b = 5$.

3. (15 points) Look at the four equations below.

$$\begin{aligned} 2 &= 2 \cdot 1 \\ 2 + 4 &= 3 \cdot 2 \\ 2 + 4 + 6 &= 4 \cdot 3 \\ 2 + 4 + 6 + 8 &= 5 \cdot 4 \end{aligned}$$

- (a) Write the next three equations in the sequence.

$$\begin{aligned} 2 + 4 + 6 + 8 + 10 &= 6 \cdot 5 \\ 2 + 4 + 6 + 8 + 10 + 12 &= 7 \cdot 6 \\ 2 + 4 + 6 + 8 + 10 + 12 + 14 &= 8 \cdot 7 \end{aligned}$$

- (b) If the four equations above correspond to $k = 1, 2, 3$, and 4, what is the n^{th} equation?

$$2 + 4 + 6 + \cdots + 2n = (n + 1)n$$

- (c) Prove by mathematical induction that the n^{th} equation is true for all integers $n \geq 1$.

Base case: $2 = (1 + 1) \cdot 1$.

Inductive step: Assume $2 + 4 + 6 + \cdots + 2n = (n + 1)n$. Then $2 + 4 + 6 + \cdots + 2n + 2(n + 1) = (n + 1)n + 2(n + 1) = (n + 2)(n + 1)$.

4. (15 points) There are 33 students in the class and sum of their ages 430 years. Is it true that one can find 20 students in the class such that the sum of their

ages is greater than 260? Explain in detail.

Solution. Arrange the ages of the 33 students in order $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_{33}$. Suppose $\sum_{i=14}^{33} a_i \leq 260$. Then $430 = 260 + 170 = \sum_{i=1}^{33} a_i \geq \sum_{i=14}^{33} a_i + 170$. It follows that $\sum_{i=1}^{13} a_i \geq 170$. Now the largest of the first 13 ages is at least the average of them which is $13\frac{1}{13}$. Therefore all the ages a_{14}, a_{15} , etc. are at least $13\frac{1}{13}$. It follows that these twenty ages have a sum of more than 261.

5. (15 points) Consider the arrangement of the letters *ANNIVERSARY*.

- (a) How many such arrangements are there?

Solution. There are $P(11; 2, 2, 2, 1, 1, 1, 1, 1) = 4,989,600$ arrangements.

- (b) When these arrangements are put in alphabetical order, what is the arrangement in position 50?

Solution. There are 60 arrangements of the type *AAEINNxxxx*, where *xxxx* refers to two *R*'s, and one each of *S*, *V*, and *Y*. Of these 60, there are 24 which look like *AAEINN Ruvxy*, then 12 more of the form *AAEINN Sxyuv*, then 12 more of the type *AAEINN Vxyuv*, so the 50th arrangement is the second of the type *AAEINN YRxyz* where $\{x, y, z\}$ is $\{R, S, V\}$. Hence the 50th arrangement is *AAEINN YRRVS*.

- (c) In alphabetical order, the arrangement *AAEINNRRSVY* comes first. What is the position of the arrangement *EAAINNRRVYS*?

Solution. There are $P(9; 2, 2, 1, 1, 1, 1, 1)$ arrangements of the type *AAxxxxxxx*, and the arrangements of the type *EAAxxxxxxx* follow right after them. Thus the answer is $P(9; 2, 2, 1, 1, 1, 1, 1) + 4 = 90,724$.

6. (15 points) An experiment consists of rolling a balanced die three times, thus obtaining an ordered triple of integers (x, y, z) . Calculate the following probabilities.

- (a) $P(x + y + z \leq 10.5)$.

Solution. For every ordered triple (x, y, z) satisfying $x + y + z \leq 10.5$ the triple $(7-x, 7-y, 7-z)$ satisfies $7-x+7-y+7-z = 21-(x+y+z) \geq 10.5$, and conversely. Thus there is one to one correspondance between the triples with sum less than 10.5 and those with sum more than 10.5. Hence $P = 1/2$.

- (b) $P(x \cdot y \cdot z \text{ is even})$.

Solution. $P(x \cdot y \cdot z \text{ is even})$ is the same as $1 - P(x \text{ is odd}) \cdot P(y \text{ is odd}) \cdot P(z \text{ is odd}) = 1 - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{7}{8}$.

- (c) $P(x + y = z)$.

Solution. $P = \frac{1}{6} \left(\frac{1+2+3+4+5}{36} \right) = 5/72$.

7. (15 points) Tilings.

- (a) How many ways can a 3×3 board be tiled by four dominos ($\square\square$) and one monomino (\square)?

Solution. The monomino must go in a corner or in the center. There are two tiling with the monomino in the center, and four tiling for each corner placement (such a tiling is determined by the placement of a domino covering the center square.) Thus the total is $2 + 4 \times 4 = 18$.

- (b) There are **four** tilings of a 3×4 board using L-shpaed triominoes ($\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}$). How many tilings of a 3×7 board are there with six L-shaped triominoes and one regular triomino ($\square\square\square$)?

Solution. $(4 + 3 \cdot 2) \cdot 2^3 = 80$.

- (c) How many tilings of a 4×4 board are there with five L-shpaed triominoes and one monomino (\square).

Solution. There is exactly one tiling for each position of the monomino. Thus, 16.

8. (15 points) Six fair identical cubical dice are rolled.

- (a) How many outcomes are possible?

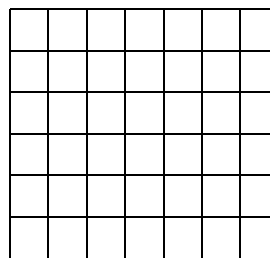
Solution. $\binom{11}{6}$.

- (b) What is the probability that three pairs are obtained? That is, what is the probability that the multiset of outcomes is u, u, v, v, w, w where u, v , and w are distinct?

Solution. $\binom{6}{3} \cdot P(6; 2, 2, 2) \div 6^6 \approx .03858$.

- (c) What is the probability that six different numbers result?

Solution. $6! \div 6^6 = 5/324 \approx .01543$



9. (20 points) Consider the 6×7 grid of unit squares.

- (a) How many rectangles are the union of one or more of the unit squares of the grid? (A 1×1 square counts as a rectangle.)

Solution. Choose the top and bottom horizontal lines in any of $\binom{7}{2} = 21$

ways and choose the left and right sides in any of $\binom{8}{2} = 28$ ways so there are $21 \cdot 28 = 588$ rectangles.

- (b) How many of the rectangles in (a) are squares?

Solution. There are $6 \cdot 7 + 5 \cdot 6 + 4 \cdot 5 + 3 \cdot 4 + 2 \cdot 3 + 1 \cdot 2 = 112$ squares.

- (c) How many of the rectangles have area 12?

Solution. There are four types to count: 2×6 , 3×4 , 4×3 and 6×2 and there are respectively, 10, 16, 15, and 6 of these for a total of 47 rectangle with area 12.

- (d) How many of the rectangles are the union of an odd number unit squares?

Solution. This is simply the number of ways to choose an odd height and an odd base: $(6 + 4 + 2) \cdot (7 + 5 + 3 + 1) = 192$.

- (e) How many of the rectangles have a perimeter that is a multiple of 3?

Solution. We want to count the number of rectangles of the sizes $a \times b$ where $a + b$ is a multiple of 3: $1 \times 2, 1 \times 5, 2 \times 1, 2 \times 4, 3 \times 3, 3 \times 6, 4 \times 2, 4 \times 5, 5 \times 1, 5 \times 4, 6 \times 3, 6 \times 2$, etc. Let $H(a \times b)$ denote the number of rectangles of size $a \times b$. Then $H(a \times b) = (7 - a)(8 - b)$. This gives rise to the sum $6 \cdot 6 + 6 \cdot 3 + 5 \cdot 7 + 5 \cdot 4 + 4 \cdot 5 + 4 \cdot 2 + 3 \cdot 6 + 3 \cdot 5 + 2 \cdot 7 + 2 \cdot 4 + 2 \cdot 1 + 1 \cdot 5 + 1 \cdot 2 = 201$.