Final Exam, Math 6105

SWIM, June 29, 2006 Your name ______________________________
Throughout this test you must **show your work**.

1. Base 5 arithmetic
   (a) Construct the addition and multiplication table for the base five digits.
   (b) Find the base 5 representations of 597 and 146.
   (c) Using the tables in (a), find the product of the two numbers in (b).
   (d) Finally compute the base 5 representation of 597×146 to check your answer.
2. Base 5 with radix point.
   (a) Find a base 5 representation of each of 3/7
   (b) Prove that your answer is correct.

3. Use the Euclidean algorithm to solve the decanting problem for containers of sizes 597 and 146; that is find integers x and y satisfying $597x + 146y = d$ where d is the GCD of 597 and 146.

4. Let $N = 2^2 \cdot 3^3 \cdot 5^5$ and $M = 2 \cdot 3 \cdot 7^4$. Find the number of positive integer divisors of each of the following.

   (a) $GCD(N, M)$

   (b) $N \cdot M$

   (c) $LCM(N, M)$

5. Consider the game of Bouton's nim with pile sizes $23, 24, 25, 27, 37$.

   (a) Find the binary representation of each pile size.

   (b) Find the binary configuration of the game. That is, write these binary numbers in a column and compute their nim sum.

   (c) Notice that the binary configuration is not balanced since the nim sum of the pile sizes is not zero. Find a move which results in a balanced binary configuration. Is there just one such move or are there several?

   (d) Suppose you made a move which balances the configuration. Assume your opponent takes seven counters from the same pile as the one from which you removed counters. What move do you make now?

6. Let $a_1 = 1$, $a_2 = 12$, and in general, a_n is the n-digit number $10a_{n-1} + u$ where $u = \begin{cases} r & \text{if } r \leq 9 \\ 19 - r & \text{if } r > 9 \end{cases}$ and r is the remainder when n is divided by 18. Thus, for example

$$\begin{aligned} a_{10} &= 10a_9 + (19 - 10) \\ &= 10 \cdot 123456789 + 9 \\ &= 1234567890 + 9 \\ &= 1234567899 \end{aligned}$$

and $a_{11} = 12345678990 + (19 - 11) = 12345678998$.

Find the first member of the sequence that is divisible by

   (a) 6

   (b) 9

   (c) 11

(d) 66
(e) 99
(f) What is the remainder when a_{2006} is divided by 66?
(g) What is the remainder when a_{2006} is divided by 99?

7. Prove that
$$1 + 3 + 3^2 + \cdots + 3^n = \frac{3^{n+1} - 1}{2}$$
for $n = 0, 1, 2, \ldots$.

8. You're playing the game $N_d(k)$ and your opponent has just left you the position $(91, 4)$. Do you have a good move? Explain. If you can make such a winning move, assume that your opponents reply is to take one more than you took on the first move. What reply would you make to that move?

9. Counting 6-card poker hands. In this problem, we assume a poker hand is a selection of 6 cards from an ordinary deck of 52 playing cards.

   (a) How many such poker hands are there?
   In all five parts below, find the number of poker hands that satisfy the given condition and are no better. For example, a three-of-a-kind hand is not counted as a hand with a pair.
   (b) A flush (all six cards in the same suit)
   (c) A full house (either a four-of-a-kind and a pair or two three-of-a-kind).
   (d) A straight (do not allow wrap-arounds, but ace can count as either high or low).
   (e) Four-of-a-kind.
   (f) Three-of-a-kind.
   (g) Three pairs.

10. Let Z denote the set of all integers. Classify each of the following functions from Z to Z as one-to-one and onto, one-to-one and not onto, onto and not one-to-one, neither onto nor one-to-one. Prove your answers.

   (a) Let $f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -2n - 1 & \text{if } n < 0 \end{cases}$
   (b) Let $f(n) = \begin{cases} n - 1 & \text{if } n \geq 1 \\ n + 1 & \text{if } n \leq 0 \end{cases}$
   (c) $f(n) = -n$
   (d) $f(n) = |n|$

11. Counting base-5 numerals. Hint: think in base 5, do not translate the problem to decimal.

    (a) How many base-5 numerals represent a positive integer less that 3125?

    (b) How many base-5 numerals have exactly 4 different digits and represent an integer less that 3125?

    (c) How many positive integers have base-5 representations that use two different digits and represent an integer less that 3125? For example, one such number is $126 = 1001_5$.

    (d) How many positive integers less that 3125 have the base-5 representation for which the rightmost digit is the sum of the digits that come before it? For example, $1102_5 = 152$ qualifies.

12. A discrete math class has 10 women and 6 men.

    (a) How many 4-element subsets does the class have?

    (b) How many ways are there to choose a committee of size 4 consisting entirely of women?

    (c) How many ways are there to choose a committee of size 4 consisting of 3 women and 1 man?

    (d) How many ways are there to choose a committee of size 4 consisting of 2 women and 2 men?

    (e) How many ways are there to choose a committee of size 4 consisting of 1 women and 3 men?

    (f) How many ways are there to choose a committee of size 4 consisting entirely of men?

13. Find a relation R on the set $S = \{1, 2, 3, 4, 5\}$ satisfying each of the following conditions. Find one relation for each part.

    (a) R_1 has exactly 7 ordered pairs members and is an equivalence relation.

    (b) R_2 has exactly 7 ordered pairs members and is reflexive and not symmetric.

    (c) R_3 is not antisymmetric, not reflexive, not transitive and $|R_3| = 7$.

    (d) R_4 is an reflexive, antisymmetric, and transitive and $|R_4| = 9$.

    (e) R_5 is transitive, not symmetric, not antisymmetric, and $|R_5| = 11$.