Directed Graphs, Boolean Matrices,and Relations

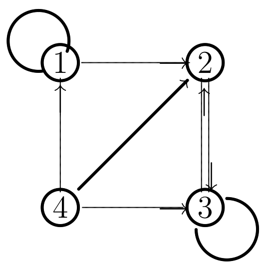
The notions of directed graphs, relations, and Boolean matrices are fundamental in computer science and discrete mathematics. These notions are quite similar or even identical, only the languages are different. In some cases the language of graph theory is preferable because of its visual nature. The graph theoretic approach to a problem enables us to think geometrically about the problem. Computers use Boolean matrices, which provide algebraic versions of the problem, and finally, logical content can be reflected easier on the language of relations.

We will discuss all three versions of the same idea in a parallel way, to stress their similarity. Another way to say this is that there are three models which all provide the same information, but in different ways. First, the digraph model emphasizes the geometry. Its useful because we can draw pictures and reason geometrically. The second is the matrix formulation. You'll see that this is the best model for computational purposes. The third model is the relational model (the ordered pair model). This representation is useful because it is most easily generalized.

I. *Directed Graphs* (*digraphs*). A graph Γ consists of a finite number of points $A = (a_1, a_2, ..., a_n)$, which are called *vertices* and a set E of directed lines between some pairs of vertices. These directed lines are called *edges* (or links, or bonds) of the graph. For example, we could let $A = \{1, 2, 3, 4\}$ and $E = \{(1,1), (1,2), (2,3), (3,2), (3,3), (4,1), (4,2), (4,3)\}$. This digraph is drawn in figure 1 below.

One can draw small circles for each element of A with its corresponding number inside and a system of edges. This provides a geometrical representation of the graph.

Fig 1.



An edge can connect the vertex a_i with itself. Such an edge is called a loop. If two vertices are connected in both directions, as a_2 and a_3 are in fig 1., we use only one line between, but with two arrows:

A typical example of a graph with two directions to each edge is a map of cities with lines between some of them indicating the existence of a direct flight between the corresponding sites. Such maps can be found in every aircraft journal.

II. For a directed graph Γ, one can construct the *Boolean* matrix M_Γ containing complete information about Γ in the numerical form.

If Γ has n vertices $A = \{a_1, ..., a_n\}$ and directed edge set E, the corresponding *adjacency* matrix M_Γ is the $n \times n$ matrix of numbers $m_{ij}, i, j, = 1, 2, ...n$ (elements or entries of M_Γ) such that

a) All entries m_{ij} are either 0 or 1.
b) $m_{ij} = 1 \Leftrightarrow$ there exists edge from a_i to a_j and $m_{ij} = 0$ otherwise.

Matrices with only 0,1 elements are called Boolean matrices. They are extremely important in logic and computer sciences. They are related to binary representation.

Example 1. For the graph Γ presented in fig 1. the corresponding Boolean matrix is

$$M_\Gamma = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Matrix M_Γ is a square matrix with as many rows and columns as Γ has vertices. Repeating what we said above, entry m_{ij} in row i and column j is 1 if there is a directed edge from the vertex a_i to a_j, and zero otherwise. Elements

$m_{11},\ m_{22}, ..., m_{nn}$ form the *main diagonal* of M_Γ. These entries describe the edges from a_i to a_i, $i = 1, 2, ..., n$, that is, the loops.

There are many useful operations on matrices (summation, multiplication, inverse matrices, etc.). In the context of graph theory and relations, we'll introduce a few new operations specific for Boolean matrices.

III. *Binary Relations.* The notion of a relation is intuitively clear. Different types of relationships between people, numbers, sets etc. are well-known. We will use the letter R (or $S, T, ...$) to indicate that two objects a and b are related. In case there are several relations in the same problem, we would say that a is R related to b to avoid the ambiguity.

aRb (or aSb, aTb, ...).

Examples.

a) *Relation of inequality* between real numbers: $aRb \Leftrightarrow\ a < b$. Here is a highly non-symmetric relation. In fact, $aRb \Rightarrow b\not{R}a$; that is if a is related to b, then b is not related to a.

b) *Relation of divisibility* for integers: aRb if a is a divisor of b, symbolically written $a|b$, i.e. $b = k \cdot a,\ k \in Z;\ a, b \in Z$.

c) *Relation of inclusion* for subsets. If $A, B \subset \mathcal{U}$ (universal set),

$$A\ R\ B \Leftrightarrow A \subset B.$$

d) Relation of equality for numbers (or sets),

$$aRb \Leftrightarrow a = b, (A\ R\ B \Longleftrightarrow A = B).$$

It is important to stress that (a, b) is an *ordered* pair of objects: just because aRb, it does not follow that bRa.

To describe a relation on the set A of objects, it is enough to give the list of all R-related pairs.

If $A = (a_1, ..., a_n)$ is a finite list of objects, the list of corresponding ordered pairs has the same structure as a sample space for the experiment of tossing a pair of dice:

$$\begin{pmatrix} (a_1, a_1)\ (a_1, a_2)...(a_1, a_n) \\ (a_2, a_1)\ (a_2, a_2)...(a_2, a_n) \\ .... \\ .... \\ (a_n, a_1)\ (a_n, a_2)...(a_n, a_n) \end{pmatrix} = A \times A$$

This quadratic $n \times n$ table contains n^2 elements. It is known as the *Cartesian product* $A \times A$ *of* A *with itself.*

Let's now mark (using some color) the subset R of this table. Pair $(a_i, a_j) \in R$ if $a_i \, R \, a_j$. To give complete information about a relation R (on the set of the objects A), it is enough to present the subset $R \subset A \times A$. We can say that this subset is itself a relation!

Briefly, the relation R (on A) is a specific subset of the set $A \times A$ of all ordered pairs of the elements of A.

It is important to be able to translate among the three models. In other words, given one of the three, you should be able to construct the other two models. It is clear how to construct for given relation R on the set A, a graph Γ_R (with vertices A), and a corresponding Boolean matrix M_R to relation R:

$$a \longrightarrow b \;\Leftrightarrow a_i \, R \, a_j \quad \Leftrightarrow m_{ij} = 1.$$

Example. Let $A = (1, 2, 3, 4)$ and $aRb \Leftrightarrow a$ is a divisor of b. The digraph (drawn without loops for convenience) is given below. All the edges are directed upwards as convention dictates.

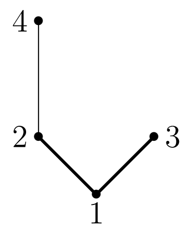


Fig. 3 (Inequality relation)

and

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operations over relations, Boolean matrices and graphs (Algebra of relations and graphs)

As in arithmetic or elementary algebra, we can deduce new more complicated formulas using the algebraic operations over the simplest formulas, we can also define the operations over graphs, Boolean matrices and relations. These operations are closely related with the set theory.

Let's start from relations. Each relation R as we know is simply a subset in the set $A \times A$ of all ordered pairs (for a given set of the objects $A = (a_1, ..., a_n)$).

1. *Compliment of R,* set $\overline{R} = (A \times A)\ R$ is, by the definition, the complimentary relation. It means, of course, that

$$a_i \, \overline{R} \, a_j \Leftrightarrow a_i \not R a_j$$

To construct the graph $\Gamma_{\overline{R}}$ we have to remove all edges of the initial graph Γ_R and draw new edges which did not exist in Γ_R.

Example. Let $A = \{1, 2, 3, 4, 5\}$ and $aRb \Leftrightarrow a+b$ is an odd number. The graph (drawn without loops as usual for graphs) is given below. Notice that each vertex in the subset $\{1, 3, 5\}$ is adjacent to each vertex in the complimentary subset $\{2, 4\}$, and that there are no edges joining pairs within either of these subsets. Such graphs are called *bipartite*. This particular graph has the name $K_{2,3}$ (the K stands for complete), which means that it has all the edges that are allowed without destroying the bipartiteness.

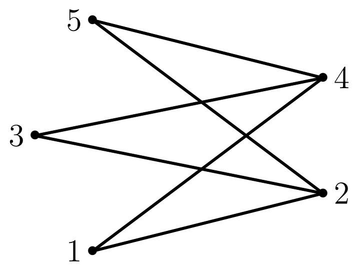


Fig. 3 (The bipartite graph $K_{2,3}$)

The compliment of R, i.e. the new relation $\overline{R}$ is: $a \, \overline{R} \, b \Leftrightarrow a+b$ is an even number. The corresponding graph $\Gamma_{\overline{R}}$ is presented in fig 4.

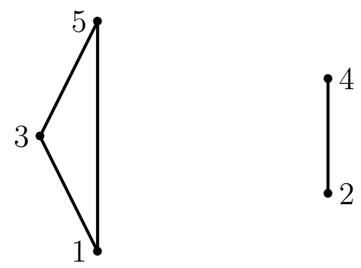


Fig 4. $\overline{K_{2,3}}$

2. The following two operations over relations are simple paraphrases of the corresponding set operations.

Let R, S be two relations on the set A, i.e. subsets in $A \times A$. Then $R \cup S$, $R \cap S$ are new relations: *union* of R and S and *intersection* of R and S.

In the language of graphs, the sense of these operations is very simple: in the graph $\Gamma_{R \cup S}$ two points a_i, a_j are connected by an edge if they are connected either in Γ_R or in Γ_S. The set of edges in $\Gamma_{R \cup S}$ is the union of the corresponding sets in Γ_R and Γ_S.

To formulate definitions of $R \cup S$ and $R \cap S$ in the language of Boolean matrices we'll start from the more general concepts. These concepts are in fact the essential elements of mathematical logic.

Let p, q be two *(Boolean) variables*, i.e. possible values for p and q are 0 or 1. We can define two new operations between p and q in the spirit of the set theory (or logic).

$$p \vee q = r \text{ (disjunction of } p \text{ and } q)$$

This operation is given by the table

p	q	$p \vee q$
0	0	0
1	0	1
0	1	1
1	1	1

Sign $\vee$ reflects, of course, the similarity between the disjunction of the Boolean variables and the unions of the sets.

Second operation (the analogue of intersection)

$p \wedge q = r$ (conjunction of p and q) is given by the table

p	q	$p \wedge q$
0	0	0
1	0	0
0	1	0
1	1	1

Using operations $\vee$ and $\wedge$ between Boolean variables one can define corresponding operations for Boolean matrices. Namely, if $A = (a_{ij},\ i,j = 1,2,...n), B = (b_{ij},\ i,j = 1,2,...n)$

then

$$A \vee B = (a_{ij} \vee b_{ij}),\ A \wedge B = (a_{ij} \wedge b_{ij})$$

For instance

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

It is clear now that Boolean matrices corresponding to union and intersection of two relations R and S have a form

$$\begin{aligned} M_{R\cup S} &= M_R \vee M_S \\ M_{R\cup S} &= M_R \wedge M_S \end{aligned}$$

3. A different type of operation on a relation R (from A to A) is the *inverse relation* R^{-1}. By the definition

$$a\ R^{-1}\ b \Leftrightarrow b\ R\ a.$$

To construct the graph $\Gamma_{R^{-1}}$ of the inverse relation it is necessary only to reverse the direction of all edges. In the language of the Boolean matrices it means that we have to interchange the positions of the elements m_{ij} and m_{ji} in the matrix M_R. This operation is known as the *transpose* of the matrix M_R (symmetrical reflection of the matrix elements with respect to the main diagonal). If A is some matrix the transpose matrix has notation M^T. For example:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \; A^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

It follows from the definition that

$$M_{R^{-1}} = (M_R)^T.$$