1. Some properties of relations. We describe below some important properties that relations might or might not have. A relation R on a set A is called

   R. *Reflexive* if $\forall x \in A, xRx$.

   S. *Symmetric* if $\forall x, y \in A, xRy \Rightarrow yRx$.

   A. *Antisymmetric* if $\forall x, y \in A, xRy$ and $yRx \Rightarrow x = y$.

   T. *Transitive* if $\forall x, y, z \in A, xRy$ and $yRz \Rightarrow xRz$.

   Now let $A = \{1, 2, 3\}$. There are $2^4 = 16$ subsets of $\{R, S, A, T\}$. Find a relation on A for each of these subsets. For example, consider the subset $\{S, A, T\}$. We seek to find a relation on $\{1, 2, 3\}$ that is symmetric, antisymmetric, and transitive, and *not* reflexive. To keep the relation from being reflexive, we must exclude one of the three ordered pairs $(1, 1), (2, 2), (3, 3)$. However, two of these could be included. So lets try $H = \{(1, 1), (2, 2)\}$. Is this symmetric? Is it transitive? Is it antisymmetric? Sketch the digraph of the relation and notice that it has just two loops. After some thought, you'll decide that H is symmetric, antisymmetric, and transitive. There are 15 other subsets of $\{R, S, A, T\}$. Find a relation for each of these, or prove that certain combinations do not exist.

2. For each relation on the real numbers $\mathbf{R}$ defined below, list the properties that it satisfies. For example, the first relation is a circle centered at the origin, so it is symmetric, but it is not reflexive, antisymmetric, or transitive.

   (a) xR_1y iff $x^2 + y^2 - 1 = 0$

   (b) xR_2y iff $xy(y - x)(y + x) = 0$

   (c) xR_3y iff $(x - \lfloor x \rfloor)(y - \lfloor y \rfloor) = 0$

   (d) xR_4y iff $\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = 1$

   (e) xR_5y iff $\lfloor x^2 \rfloor + \lfloor y^2 \rfloor = 1$

   (f) xR_6y iff $x - y = \lfloor x - y \rfloor$.

   (g) xR_7y iff $2|(\lfloor x \rfloor - \lfloor y \rfloor)$

3. Let $\mathbf{R}$ and $\mathbf{T}$ be the relations on $A = \{1, 2, 3, 4, 5, 6\}$ defined as follows:

$$a\mathbf{R}b \iff a - b \text{ is divisible by } 2$$
$$a\mathbf{T}b \iff a \leq b$$

   Construct the Boolean matrices and the digraph for each of the relations $\mathbf{R}, \mathbf{T}, \overline{\mathbf{T}}, \mathbf{R^{-1}}, \mathbf{R} \cup \mathbf{T}, \mathbf{R} \cap \mathbf{T}$ and determine which of the properties listed in problem 1 are satisfied by each of these. See lecture 9 for the definitions of $\overline{\mathbf{R}}$ and $\mathbf{R}^{-1}$.

4. Let N be an n element set. Then there are 2^{n^2} relations on N. Thus there are $2^9 = 512$ relations on a three element set. Use the matrix model or the digraph model to find the number of relations on the set $S = \{1, 2, 3\}$ that are
   (a) reflexive (R)
   (b) symmetric (S)
   (c) antisymmetric (A)
   (d) transitive (T)
   (e) equivalence relations (RST)
   (f) partial orderings (RAT)

5. Define a relation R on Z as follows: $aRb \Leftrightarrow b - a$ is a multiple of 7. This is another way of saying $aRb \Leftrightarrow b \equiv a (\text{mod } 7)$. Prove that R is an equivalence relation on Z. What are the cells of the partition determined by R? Define the sum and the product of cells and construct the two arithmetic tables.

6. A positive six-digit integer $d_1 d_2 \ldots d_6$, where $d_1 \neq 0$ is said to be *3-special* if each of the numbers $10d_1 + d_2, 10d_2 + d_3, \ldots, 10d_5 + d_6$ is a multiple of 3. How many 3-special numbers there?

7. A positive ten-digit integer $d_1 d_2 \ldots d_{10}$, where $d_1 \neq 0$ is said to be 7-*special* if each of the numbers $10d_1 + d_2, 10d_2 + d_3, \ldots, 10d_9 + d_{10}$ is a multiple of 7. How many 7-special numbers there?