1. Pick two numbers a and b. Let $a_1 = a$ and $a_2 = b$. Then define a_{n+2} by $a_{n+2} = \frac{a_{n+1}+1}{a_n}$. In other words, the first integer picked is the first number and the second is the second. The third is the quotient of 1 plus the second and the first. Then get the fourth by doing the same thing with the third and second. Continue the process, getting the fifth, sixth, etc. For example, suppose the first number is 3 and the second is 5. Then the third would be $\frac{5+1}{3} = 2$ and the fourth would be $\frac{2+1}{5} = \frac{3}{5}$, and the fifth is $\frac{\frac{3}{5}+1}{2} = \frac{4}{5}$. Now compute the sixth number in the sequence:
$$\frac{\frac{4}{5}+1}{\frac{3}{5}} = 3$$
and the one following that is:
$$\frac{3+1}{\frac{4}{5}} = 5,$$
so you can see that the sequence starts all over again. Such sequences are called *periodic*. This one has period 5 because the sixth term is the same as the first, etc. You repeat the process with two other initial picks. Again you get periodicity. Do you always get a periodic sequence?

2. Prove that all the numbers $x_n = n\left(n^2+5\right) \qquad n = 0, 1, 2, \ldots$ are divisible by 6.

3. Let
$$a_n = \frac{1}{n\,(n+1)}, \qquad n = 1, 2, \ldots$$
and $\qquad S_n = a_1 + a_2 + \cdots + a_n. \qquad$ Prove that
$$S_n = \frac{n}{n+1}$$
Find $\lim\limits_{n\to\infty} S_n$.

4. Let $a_n = 3n^2 - 3n + 1$. Prove that $S_n = a_1 + a_2 + \cdots + a_n = n^3$.

5. Prove that for some integer n_0, $3^n > 5n,$ for all $n \geq n_0$. Find the minimal n_0.

6. Let f_n be the Fibonacci sequence: $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$, for $n = 2, 3, \ldots$. Prove that $f_0 + f_1 + \cdots + f_n = f_{n+2} - 1$ for all $n \geq 0$.

7. Let $a_n = 3a_{n-1} - 2a_{n-2}$, $a_1 = 1$, $a_2 = 3$. Guess the formula for a_n and prove the result by mathematical induction.

8. Let $a_1 = \sqrt{2}$, $a_2 = \sqrt{2+\sqrt{2}}$, $\cdots$

$$a_n = \underbrace{\overset{n \text{ radicals}}{\sqrt{2+\sqrt{2+\cdots\sqrt{2}}}}}$$

Find the first-order recursive relation for the sequence a_n. The rest of this problem is not part of the homework for the Fall 2000 class, but is expected for all classes thereafter. Prove each of the following about the sequence.

   (a) The sequence is *bounded above.* That is, there exists a number B such that $a_n \le B$ for all $n \ge 1$.

   (b) The sequence is *non-decreasing.* That is, $a_n \le a_{n+1}$ for all $n \ge 1$.

   (c) The two conditions above together imply that the sequence *converges.* That is $\lim\limits_{n\to\infty} a_n$ exists. Find the limit.