Final Exam, Math 6105

July 29, 2004 Your name ______________________________

Throughout this test you must **show your work**.

1. Use the repeated subtraction method to find the base 4 representation of each of the following numbers
   (a) 93
   Solution: $1131_4 = 93$.
   (b) 17.25
   Solution: $101.1_4 = 17.25$.
2. Use the method of repeated multiplication to find a base 4 representation of each of the following numbers
   (a) 0.275
   Solution: $0.10\overline{12}_4 = 0.275$.
   (b) 29/64
   Solution: $0.131_4 = 29/64$
3. Find the base -4 representation of each of the following numbers
   (a) 93
   Solution: $13211_{-4} = 93$.
   (b) 17.25
   Solution: $102.3_{-4} = 17.25$.
4. Find the Fibonacci representation of each of the following numbers
   (a) 93
   Solution: $93 = 89 + 3 + 1 = 1000000101_f$.
   (b) 180
   Solution: $180 = 144 + 34 + 2 = 10010000010_f$.
5. You're playing the game $N_d(k)$ and your opponent has just left you the position $(93, 6)$. Do you have a good move? Explain.

   Solution: You can assure a win by moving to either $(89, 8)$ or to $(92, 2)$.
6. Consider the game of Bouton's nim with pile sizes $19, 24, 25, 27, 35$.
   (a) Find the binary representation of each pile size.
   Solution: $19 = 10011_2$; $24 = 11000_2$; $25 = 11001_2$; $27 = 11011_2$; and $35 = 100011_2$.

(b) Find the binary configuration of the game. That is, write these binary numbers in a column and compute their nim sum.

Solution:

$$\begin{array}{rcrrrrrr} 19 & = & & 1 & 0 & 0 & 1 & 1 \\ 24 & = & & 1 & 1 & 0 & 0 & 0 \\ 25 & = & & 1 & 1 & 0 & 0 & 1 \\ 27 & = & & 1 & 1 & 0 & 1 & 1 \\ 35 & = & \underline{1} & \underline{0} & \underline{0} & \underline{0} & \underline{1} & \underline{1} \\ & & 1 & 0 & 1 & 0 & 1 & 0 \end{array}$$

(c) Notice that the binary configuration is not balanced since the nim sum of the pile sizes is not zero. Find a move which results in a balanced binary configuration. Is there just one such move or are there several?

Solution: There is just one winning move, and $(19, 24, 25, 27, 35). \mapsto (19, 24, 25, 27, 9)$.

(d) Suppose you made a move which balances the configuration. Assume your opponent takes one counter from the same pile as the one from which you removed counters. What move do you make now?

Solution: The three winning moves are $(19, 24, 25, 27, 8) \mapsto (18, 24, 25, 27, 8)$; $(19, 24, 25, 27, 8) \mapsto (19, 24, 24, 27, 8)$; and $(19, 24, 25, 27, 8) \mapsto (19, 24, 25, 26, 8)$

7. Find the number of positive integer divisors of the number 10!. Explain how you got your answer.

Solution: First factor 10! to get $10! = 7 \cdot 5^2 \cdot 3^4 \cdot 2^8$. Therefore, by the divisor counting formula, $|D_{10!}| = 2 \cdot 3 \cdot 5 \cdot 9 = 270$.

8. Find the remainder when each of the following numbers is divided by 6.

(a) $N =$ 123,456,789,101,112

Solution: Since N is both even and a multiple of 3, it follows that $N \equiv 0 \pmod 6$.

(b) $N = 5^{2004}$

Solution: First note that $5 \equiv -1 \pmod 6$. It follows that $5^{2004} \equiv (-1)^{2004} = 1 \pmod 6$.

(c) $N = 3^{2001} \cdot 5^{2004} \cdot 7^{2005}$

Solution: N is an odd multiple of 3. Therefore $N \equiv 3 \pmod 6$.

9. How many of the first 1000 positive integers have an odd number of positive integer divisors? Explain your work.

   Solution: We know that a number has an odd number of divisors precisely when it is a perfect square. There are 31 perfect squares in the set $\{1, 2, 3, \ldots, 1000\}$.

10. Look at the four equations below.

$$\begin{array}{lcl} 2 & = & 2 \cdot 1 \\ 2+4 & = & 3 \cdot 2 \\ 2+4+6 & = & 4 \cdot 3 \\ 2+4+6+8 & = & 5 \cdot 4 \end{array}$$

   (a) Write the next three equations in the sequence.

   Solution:

$$\begin{array}{lcl} 2+4+6+8+10 & = & 6 \cdot 5 \\ 2+4+6+8+10+12 & = & 7 \cdot 6 \\ 2+4+6+8+10+12+14 & = & 8 \cdot 7 \end{array}$$

   (b) If the four equations above correspond to $k = 1, 2, 3,$ and 4, what is the n^{th} equation?

   Solution:

$$2+4+6+8\ldots+2n \quad = \quad (n+1) \cdot n$$

   (c) Prove by mathematical induction that the n^{th} equation is true for all integers $n \geq 1$.

   Solution: The base case: $2 = (1+1) \cdot 1$. Assume $P(n) : 2+4+6+8\ldots+2n = (n+1) \cdot n$. To prove $P(n+1) : 2+4+6+8\cdots+2n+2(n+1) = (n+2) \cdot (n+1)$, start with the left side and replace the sum of the first n terms with the right side of $P(n)$. Thus $2+4+6+8\cdots+2n+2(n+1) = (2+4+\cdots 2n)+2(n+1) = (n+1) \cdot n + 2(n+1) = (n+1)(n+2)$, which is the right side of $P(n+1)$. By mathematical induction, it follows that $P(n)$ is true for all $n \geq 1$.

11. Solve the decanting problem for containers of sizes 138 and 147; that is find integers x and y satisfying $138x + 147y = d$ where d is the GCD of 138 and 147.

   Solution: By repeated division, $x = 16$ and $y = -15$.

12. Find a relation R on the set $S = \{1, 2, 3, 4\}$ satisfying each of the following conditions. Find one relation for each part.

(a) R_1 has exactly 5 ordered pairs members and is transitive.

Solution: Of course there are many correct answers. One is $R_1 = \{(1,1),(2,2),(3,3),(4,4),(1,2)\}$.

(b) R_2 has exactly 5 ordered pairs members and is not transitive.

Solution: Again there are many correct answers. One is $R_1 = \{(1,1),(4,4),(3,3),(2,1),(1,2)\}$.

(c) R_3 is symmetric and has exactly 6 ordered pairs members.

Solution: Again there are many correct answers. One is $R_1 = \{(1,1),(2,2),(4,3),(3,4),(2,1),(1,2)\}$.

(d) R_4 is an equivalence relation with exactly 6 ordered pairs members.

Solution: The partition $\{1,2\},\{3\},\{4\}$ induces an equivalence relation with $4+1+1=6$ ordered pairs.

(e) R_5 is a partially ordered set with exactly 9 ordered pairs members.

Solution: Consider the poset with 4 maximal, 3 below 4, and 1 and 2 tied below 3. The relation has 9 members.