

Final Exam, Math 6105

SWIM, July 20, 2007

Your name _____

Throughout this test you must **show your work**.

1. (10 points) Euclidean Algorithm
 - (a) Does the equation $119x + 399y = 14$ have an integer solution in x and y ?
Solve or explain why there is no solution.

 - (b) Does the equation $119x + 399y = 22$ have an integer solution in x and y ?
Solve or explain why there is no solution.

2. **The Subtraction Game.** (10 points) In the Subtraction Game, two players start with some positive integers written on a board. The first player must find a pair of numbers whose positive difference is not already written on the board. Then he writes this new number on the board. At each stage, the next player finds a positive difference between two numbers on the board that is not already written on the board and writes it on the board. The first player who cannot find a new positive difference loses. For the set of numbers listed below, decide how many numbers will be on the board at the end of the game. Use this information to state whether you would like to move first or not (in order to win). 108, 111, 123

3. (10 points) A check is written for x dollars and y cents, both x and y two-digit numbers. In error it is cashed for y dollars and x cents, the incorrect amount exceeding the correct amount by \$18.81. Find a possible value for x and y .

4. (10 points) Is it possible to decomposed a 10×10 square is into exactly 53 squares all of integer sizes?

5. (15 points) Is there a six-digit number \underline{abcdef} which is one-sixth of the number obtained by moving the first three digits to the other end? Symbolically, $6 \cdot \underline{abcdef} = \underline{defabc}$.
6. (20 points) Recall the two functions *floor* ($\lfloor \rfloor$) and *fractional part* ($\langle \rangle$), defined by $\lfloor x \rfloor$ is the largest integer that is less than or equal to x , and $\langle x \rangle = x - \lfloor x \rfloor$.
- (a) Is there a rational number r whose fractional part is the same as the fractional part of its reciprocal?
- (b) Find a number x such that (a) $1 < x < 2$ and (b) $\langle x \rangle = \langle 1/x \rangle$.
- (c) Find a number x such that (a) $2 < x < 3$ and (b) $\langle x \rangle = \langle 1/x \rangle$.
7. (12 points) The sum of a three digit integer n and the number n obtained by moving the hundreds digit of n to the right end is 475. Find n .
8. (10 points) Find the base 7 representation of the **decimal** 2008. Then interpret your base seven numeral as a sum of multiples of powers of 7 to get the decimal representation of your number.
9. (12 points) Find the base -7 representation of each of the following:
- (a) -35
- (b) 35
- (c) 102
- (d) -102
10. (10 points) **Finding the unknown digit.**
 Let $N = \underline{abcde}$ denote the five digit number with digits a, b, c, d, e and $a \neq 0$. Let $N' = \underline{edcba}$ denote the reverse of N . Suppose that $N > N'$ and that $N - N' = 270x7$ where d is a digit. What is x ? Explain your answer.
11. (15 points) The Lucas numbers are defined as follows. $L_1 = 2$ and $L_2 = 1$, and for $n \geq 3$, $L_n = L_{n-1} + L_{n-2}$. Find the units digit of L_{2007} .
12. (15 points) You're playing Bouton's Nim with a partner who knows the theory. But you can decide who plays first. In each case below, tell whether you'd choose to play first or not and say why. If you decide to play first, find a position you would move to in order to win the game.
- (a) $(16, 17, 4)$
- (b) $(16, 17, 4, 5)$

(c) (32, 29, 16, 17, 4, 5)

13. (15 points) Consider the number $M = 0.0123456789101112\dots$ obtained by writing the nonnegative integers in order next to one another after the decimal point. In this problem all numbers are in their usual decimal notation.

(a) What is the 26th digit to the right of the decimal point of M ?

(b) What is the 206th digit to the right of the decimal point of M ?

(c) What is the 2007th digit to the right of the decimal point of M ?

14. (12 points) Find the sum of elements of each of the arithmetic sequences of numbers below. Note that item (a) could be phrased ‘what is the sum of all 4-digit multiples of 7?’

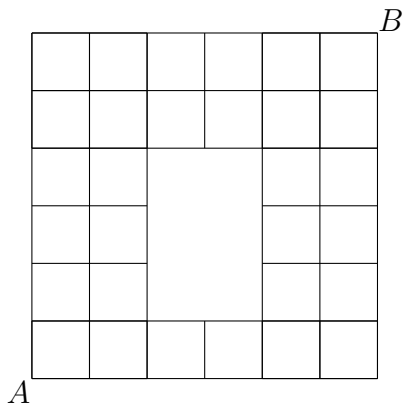
(a) $1001 + 1008 + 1015 + 1022 + 1029 + \dots + 9996$

(b) $1001 + 1012 + 1023 + 1034 + 1045 + \dots + 9999$

(c) $1001 + 1014 + 1027 + 1040 + 1054 + \dots + 9997$

(d) $48 + 55 + 62 + 69 + 76 + \dots + 699$

15. (20 points) Consider the ‘annular’ grid of unit squares. You can think of this as a design for a downtown area with a park in the middle.



(a) How many square subregions (of all sizes) does the figure have?

(b) How many rectangles have sides determined by the grid lines below?

(c) How many paths of length 12 are there from the lower left corner A to the upper right corner B ?

16. (15 points) Rational and irrational numbers.

- (a) Is the number $M = 0.0123456789101112\dots$, obtained by writing the decimal digits of 0, then 1, etc. rational or irrational? Elaborate on your answer.
- (b) Is the number $M = 0.0123123\overline{123}$ rational or irrational? Elaborate on your answer. (Included after the test: elaborate here means find integers m and n such that $M = m/n$.)
17. (12 points) Among 18 students in a room, 7 study mathematics, 10 study science, and 10 study computer programming. Also, 3 study mathematics and science, 4 study mathematics and computer programming, and 5 study science and computer programming. We know that 1 student studies all three subjects. How many of these students study none of the three subjects?
18. (20 points) Notice that

$$\frac{1}{1 \cdot 2} = \frac{1}{2} \tag{1}$$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{2}{3} \tag{2}$$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{3}{4} \tag{3}$$

- (a) List the next two equations suggested by the pattern.
- (b) Given that the three equations above are the 1st, 2nd, and 3rd, write the n^{th} equation of the sequence.
- (c) Use mathematical induction to prove that the n^{th} equation is true for all positive integer values of n .
19. (12 points) Let $N = 2^2 \cdot 3^3 \cdot 5^5$ and $M = 2^4 \cdot 3 \cdot 5 \cdot 7^3$.
- (a) What is the greatest common divisor of M and N ?
- (b) What is the least common multiple of M and N ?
- (c) How many (positive integer) divisors does N have?
- (d) How many of the numbers $1, 2, 3, 4, \dots, 1000$ have an odd number of divisors?

20. (35 points) Let $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the universal set. Let D denote the set of all four-digit numbers that can be built using the elements of S as digits and allowing repetition of digits.
- (a) Find the number of four element subsets of S .

 - (b) How many of the four element subsets of S have two odd and two even number members?

 - (c) What is $|D|$? In other words, how many four-digit numbers are there?

 - (d) How many elements of D have four different digits?

 - (e) How many elements of D have exactly three different digits?

 - (f) How many members of D have a sum of digits equal to 8? IE, what is $|\{n \in D \mid S(n) = 8\}|$?

 - (g) How many even numbers belong to D ?