UNC Charlotte 2007 Comprehensive Contest

March 5, 2007

1. Maya deposited 1000 dollars at 6% interest compounded annually. What is the number of dollars in the account after four years?

   (A) \$1258.47 **(B)** \$1260.18 **(C)** \$1262.48

   (D) \$1263.76 **(E)** \$1264.87

2. What is the area (in square units) of the region of the first quadrant defined by $18 \leq x + y \leq 20$?

   (A) 36 **(B)** 38 **(C)** 40 **(D)** 42 **(E)** 44

3. How many four-digit numbers between 6000 and 7000 are there for which the thousands digits equal the sum of the other three digits?

   (A) 20 **(B)** 22 **(C)** 24 **(D)** 26 **(E)** 28

4. How many positive two-digit integers have an odd number of positive divisors?

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

5. If x is positive, what is the least value of $x + \frac{9}{x}$?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 6

6. The area of an annular region bounded by two concentric circles is 5π square centimeters. The difference between the radii of the circles is one centimeter. What is the radius of the smaller circle, in centimeters?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 6

7. If we divide 344 by d the remainder is 3, and if we divide 715 by d the remainder is 2. Which of the following is true about d?

   (A) $10 \leq d \leq 19$ **(B)** $20 \leq d \leq 29$ **(C)** $30 \leq d \leq 39$

   (D) $40 \leq d \leq 49$ **(E)** $50 \leq d \leq 59$

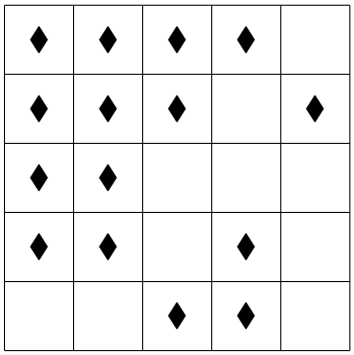
8. The sum $a + b$, the product $a \cdot b$ and the difference of squares $a^2 - b^2$ of two positive numbers a and b is the same nonzero number. What is b?

   (A) 1 **(B)** $\frac{1+\sqrt{5}}{2}$ **(C)** $\sqrt{3}$ **(D)** $\frac{7-\sqrt{5}}{2}$ **(E)** 8

9. An athlete covers three consecutive miles by swimming the first, running the second and cycling the third. He runs twice as fast as he swims and cycles one and a half times as fast as he runs. He takes ten minutes longer than he would do if he cycled the whole three miles. How many minutes does he take?

   (A) 16 **(B)** 22 **(C)** 30 **(D)** 46 **(E)** 70

10. What is the fewest crickets that must hop to new locations so that each row and each column has three crickets? Crickets can jump from any square to any other square.



   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

11. A quadrilateral $ABCD$ has vertices with coordinates $A\,(0,0)$, $B\,(6,0)$, $C\,(5,4)$, $D\,(3,6)$. What is its area?

   (A) 18 **(B)** 19 **(C)** 20 **(D)** 21 **(E)** 22

12. There are six ways to insert two multiplication signs in the string 33223 keeping the digits in the same order. For example, $33 \cdot 22 \cdot 3 = 2178$. If one puts these numbers in order from least to largest, in which place does 2007 occur?

   (A) 1^{st} **(B)** 2^{nd} **(C)** 3^{rd} **(D)** 4^{th} **(E)** 5^{th}

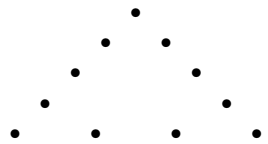
13. Given are two parallel lines of distance 1 apart and a circle of radius 2. The circle is tangent to one of the lines and cuts the other line. The area of the circular cap between the two parallel lines is $a\frac{\pi}{3} - b\sqrt{3}$. Find the sum $a + b$ of the two integers a and b.

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

14. Exactly one four digit number N satisfies $9 \cdot N = \overline{N}$ where $\overline{N}$ is obtained from N by reversing the digits. What is the sum of the digits of N?

(A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 20

15. A triangular grid of 11 points is given. How many triangles have all three vertices among the 11 points?



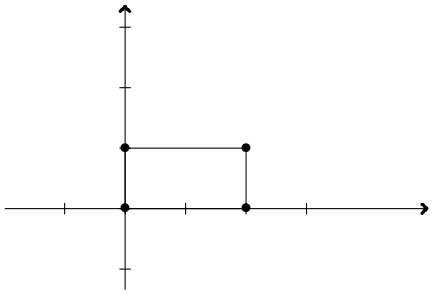
(A) 140 **(B)** 141 **(C)** 142 **(D)** 150 **(E)** 165

16. Let $f(x) = \frac{x-1}{x+1}$ and let $f^{(n)}(x)$ denote the n-fold composition of f with itself. That is, $f^1(x) = f(x)$ and $f^{(n)}(x) = f\left(f^{(n-1)}(x)\right)$. Which of the following is $f^{(2007)}(x)$?

(A) $-\dfrac{1}{x}$ **(B)** $-\dfrac{x+1}{x-1}$ **(C)** $\dfrac{1}{x}$ **(D)** $\dfrac{1-x}{1+x}$ **(E)** $\dfrac{x-1}{x+1}$

17. A point (x, y) is selected at random from the rectangular region shown. What is the probability that $x < y$?

(A) $1/5$ **(B)** $1/4$ **(C)** $1/3$ **(D)** $1/2$ **(E)** $2/3$



18. The number 240,240 can be expressed as a product of k consecutive integers. A possible value of k is

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

19. Let A be the area of a triangle with sides $5, 5$, and 8, and let B denote the area of a triangle with sides $5, 5$, and 6. Which of the following is true.

(A) $A < B < 12$ **(B)** $B < A < 12$ **(C)** $A = B$

(D) $12 < A < B$ **(E)** $12 < B < A$

20. Suppose a, b and c are real numbers for which

$$\frac{a}{b} > 1 \text{ and } \frac{a}{c} < -1.$$

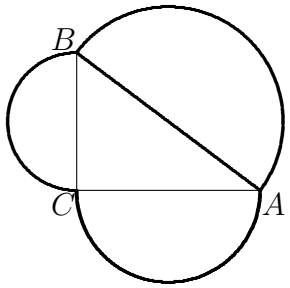
Which of the following must be correct?

(A) $a + b - c > 0$ **(B)** $a > b$ **(C)** $(a - c)(b - c) > 0$

(D) $a - b + c > 0$ **(E)** $abc > 0$

21. A right triangle ABC is given. Semicircles are constructed with the sides of the triangle as diameters, as shown below. Suppose the area of the largest semicircle is 36 and the area of the smallest one is 16. What is the area of the other one?

    (A) 20 **(B)** 24 **(C)** 25 **(D)** 26 **(E)** 30



22. A standard deck of 52 cards contains 13 hearts. Twenty six cards have already been dealt, eight of which are hearts. If you are dealt 13 of the remaining cards, what is the probability that you will get exactly 2 of the remaining 5 hearts? (Round your answer.)

    (A) 22% **(B)** 26% **(C)** 30% **(D)** 34% **(E)** 38%

23. Find the value of the expression

$$S = 1! \cdot 3 - 2! \cdot 4 + 3! \cdot 5 - 4! \cdot 6 + \ldots - 2006! \cdot 2008 + 2007!$$

    (A) -2007 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2007

24. You bought a big cake for a party and expect 10 or 11 people to come. What is the minimal number of pieces (perhaps of different sizes) you need to divide the cake evenly if exactly 10 guests attend and also evenly among 11 guests?

    (A) 11 **(B)** 20 **(C)** 30 **(D)** 55 **(E)** 110