UNC Charlotte 2007 Comprehensive Contest

March 5, 2007

1. Maya deposited 1000 dollars at 6% interest compounded annually. What is the number of dollars in the account after four years?

   (A) \$1258.47 **(B)** \$1260.18 **(C)** \$1262.48

   (D) \$1263.76 **(E)** \$1264.87

   Solution: C. The amount in the account is given by $A = P(1+r/n)^{nt}$, where P is the principal, r is the annual rate, n is the number of times per year that interest is compounded, and t is the time in years. Therefore, the amount in the account after four years is $1000(1 + .06/1)^4 \approx 1262.48$.

2. What is the area (in square units) of the region of the first quadrant defined by $18 \leq x + y \leq 20$?

   (A) 36 **(B)** 38 **(C)** 40 **(D)** 42 **(E)** 44

   Solution: B. In the first quadrant, the inequalities $x + y \leq 20$ and $x + y \leq 18$ define similar right triangles with areas $20^2/2$ and $18^2/2$. The area of the region in question is therefore $(20^2 - 18^2) \div 2 = 38$ square units.

   Alternatively, we can also find the area as the area of a trapezoid. The parallel sides have lengths $20\sqrt{2}$ and $18\sqrt{2}$ and the height of the trapezoid is $\sqrt{2}$. Thus the area is $(38\sqrt{2}/2) \cdot \sqrt{2} = 38$.

3. How many four-digit numbers between 6000 and 7000 are there for which the thousands digits equal the sum of the other three digits?

   (A) 20 **(B)** 22 **(C)** 24 **(D)** 26 **(E)** 28

   Solution: E. The question is equivalent to ‘how many three digit numbers $\underline{abc}$ (0 allowed as the hundreds digit) satisfy $a + b + c = 6$. When $a = 6$, there is only one way to do this: $b = c = 0$. When $a = 5$, there are two ways: either $b = 1$ and $c = 0$ or $c = 1$ and $b = 0$. When $a = 4$, there are three ways: $(b, c) = (0, 2), (1, 1)$, or $(2, 0)$. Continuing in this way, we find that there are $1 + 2 + 3 + \ldots + 7 = 28$ ways to build the number.

   Alternatively, there are three ways to choose three different digits that sum to 6: $1+2+3$, $0+2+4$ and $0+1+5$. Since the order counts, this gives 18 ways. There are three ways to choose two digits the same and a different third digit: $1 + 1 + 4$, $0 + 3 + 3$ and $0 + 0 + 6$. Since two of the digits are the same, there are 9 ways of ordering these. Finally, there is one sum, $2 + 2 + 2$, where all three digits are the same. So the total number of ways is $18 + 9 + 1 = 28$.

4. How many positive two-digit integers have an odd number of positive divisors?

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

   Solution: D. Since each divisor d of a number D can be paired with a divisor D/d, only the perfect squares can have an odd number of divisors. There are 6 perfect squares between 10 and 99.

5. If x is positive, what is the least value of $x + \frac{9}{x}$?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 6

   Solution: E. Note: $x+\frac{9}{x} = \frac{1}{x}(x^2+9) = \frac{1}{x}(x^2-6x+9)+6 = \frac{1}{x}(x-3)^2+6 \geq 6$.

6. The area of an annular region bounded by two concentric circles is 5π square centimeters. The difference between the radii of the circles is one centimeter. What is the radius of the smaller circle, in centimeters?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 6

   Solution: B. If r is the radius of the smaller circle, the area **of the annular region** is $(r+1)^2\pi - r^2\pi = (2r+1)\pi$. The solution of the equation $2r+1 = 5$ is $r = 2$.

   Alternatively, let s be the radius of the larger circle Then $5\pi = s^2\pi - r^2\pi = (s-r)(s+r)\pi$. Since $s - r = 1$, $s + r = 5$. So $s = 3$ and $r = 2$.

7. If we divide 344 by d the remainder is 3, and if we divide 715 by d the remainder is 2. Which of the following is true about d?

   (A) $10 \leq d \leq 19$ **(B)** $20 \leq d \leq 29$ **(C)** $30 \leq d \leq 39$

   (D) $40 \leq d \leq 49$ **(E)** $50 \leq d \leq 59$

   Solution: C. The number d divides $341 = 11 \cdot 31$ and $713 = 23 \cdot 31$. The only common divisors are 1 and 31. Since we get nonzero remainders, $d = 31$.

8. The sum $a + b$, the product $a \cdot b$ and the difference of squares $a^2 - b^2$ of two positive numbers a and b is the same nonzero number. What is b?

   (A) 1 **(B)** $\frac{1+\sqrt{5}}{2}$ **(C)** $\sqrt{3}$ **(D)** $\frac{7-\sqrt{5}}{2}$ **(E)** 8

   Solution: B. Since $a + b = a^2 - b^2 = (a + b) \cdot (a - b)$, dividing both sides by $a + b \neq 0$ yields $1 = a - b$, so $a = b + 1$. Substituting this into $a + b = a \cdot b$ we get $2b + 1 = (b + 1) \cdot b$. The only positive solution of this quadratic equation is

$$b = \frac{1 + \sqrt{5}}{2}.$$

9. An athlete covers three consecutive miles by swimming the first, running the second and cycling the third. He runs twice as fast as he swims and cycles one and a half times as fast as he runs. He takes ten minutes longer than he would do if he cycled the whole three miles. How many minutes does he take?

   (A) 16 **(B)** 22 **(C)** 30 **(D)** 46 **(E)** 70

   Solution: B. Let S denote the time in hours required to swim a mile. Then the time required to run a mile is $\frac{S}{2}$, and the time needed to cycle a mile is $\frac{S}{3}$. It follows that $S + \frac{1}{2}S + \frac{1}{3}S - \frac{1}{6} = 3(\frac{1}{3}S)$, so $S = \frac{1}{5}$, and $S + \frac{1}{2}S + \frac{1}{3}S = \frac{11}{6}\frac{1}{5} = \frac{11}{30}$ hours, which is 22 minutes.

   Alternatively, let s be the swimming speed, r the running speed and c the cycling speed in miles per minute. Then his total time for the three miles is $t = (1/s) + (1/r) + (1/c) = 10 + (3/c)$. Also, $r = 2s$ and $c = 1.5r$. So $c = 3s$, $1/s = 3/c$ and $1/r = 3/2c$. Subbing in yields $5/2c = 10$ and $1/c = 4$. So the total time is $10 + 3(4) = 22$ minutes.

10. What is the fewest crickets that must hop to new locations so that each row and each column has three crickets? Crickets can jump from any square to any other square.

♦	♦	♦	♦	
♦	♦	♦		♦
♦	♦			
♦	♦		♦	
		♦	♦	

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

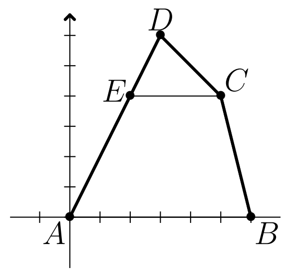
Solution: C. Notice that two rows have four crickets, so at least two crickets must move. The pair of crickets at $(1,5)$ and $(2,4)$ on the main diagonal can be moved to $(5,3)$ and $(5,1)$ as shown.

	♦	♦	♦	
♦		♦		♦
♦	♦			♦
♦	♦		♦	
		♦	♦	♦

11. A quadrilateral $ABCD$ has vertices with coordinates $A\,(0,0)$, $B\,(6,0)$, $C\,(5,4)$, $D\,(3,6)$. What is its area?

(A) 18 **(B)** 19 **(C)** 20 **(D)** 21 **(E)** 22

Solution: D. The point $E = (2,4)$ lies on AD. The line segment from $(2,4)$ to $(5,4)$ divides the quadrilateral into a trapezoid with area $\left(\frac{6+3}{2}\right)4$ and a triangle with area $\frac{1}{2}(3)(2)$. The total area is 21.



12. There are six ways to insert two multiplication signs in the string 33223 keeping the digits in the same order. For example, $33 \cdot 22 \cdot 3 = 2178$. If one puts these numbers in order from least to largest, in which place does 2007 occur?

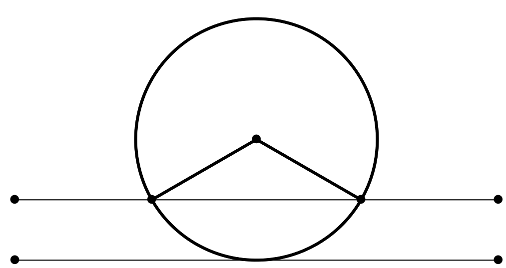
(A) 1^{st} **(B)** 2^{nd} **(C)** 3^{rd} **(D)** 4^{th} **(E)** 5^{th}

Solution: C. There are just six such numbers. The six possible choices are $3 \cdot 3 \cdot 223 = 2007$, $3 \cdot 32 \cdot 23 = 2208$, $3 \cdot 322 \cdot 3 = 2898$, $33 \cdot 2 \cdot 23 = 1518$, $33 \cdot 22 \cdot 3 = 2178$, and $332 \cdot 2 \cdot 3 = 1992$. So the number 2007 is third in the list.

13. Given are two parallel lines of distance 1 apart and a circle of radius 2. The circle is tangent to one of the lines and cuts the other line. The area of the circular cap between the two parallel lines is $a\frac{\pi}{3} - b\sqrt{3}$. Find the sum $a + b$ of the two integers a and b.

    (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Solution: C. The cap is a segment minus a triangle. The central angle of the segment is $120°$. The circular segment covers one third of the circle and hence has area $\frac{4\pi}{3}$. Hence $a = 4$. The triangle is isosceles with height 1 and two congruent sides of length 2. Its third side has length $2\sqrt{3}$ and hence its area is $\sqrt{3}$. Hence $b = 1$ and $a + b = 5$.



14. Exactly one four digit number N satisfies $9 \cdot N = \overline{N}$ where $\overline{N}$ is obtained from N by reversing the digits. What is the sum of the digits of N?

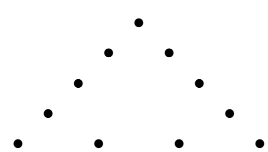
    (A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 20

    Solution: D. Symbolically, $9 \cdot abcd = dcba$. In other words, $9\,(1000a + 100b + 10c + d) = 1000d + 100c + 10b + a$. Thus $9000a + 900b + 90c + 9d = 1000d + 100c + 10b + a$ and $8999a + 890b = 10c + 991d$, so we must have $a = 1$ so $d = 9$. Then $900b + 90c + 81 = 100c + 10b + 1$, so $890b + 80 = 10c$. Dividing by 10 yields $89b + 8 = c$ from which it follows that so $b = 0$ and $c = 8$. So the sum of the digits is $1 + 0 + 8 + 9 = 18$.

    Alternatively, if the number is $N = \underline{abcd}$, then a must be 1, d must be 9 and b is at most 1 since $9 \cdot 1200 > 10,000$. Since $9 \cdot 9 = 81$, $b = 0$ implies the units digit of $9 \cdot c$ is 2 so $c = 8$, and $b = 1$ implies the units digit of $9 \cdot c$ is 3 so $c = 7$. Since $9 \cdot 1179$ is a five digit number, $N = 1089$. So the sum of the digits is 18 (the same as the sum for 1179).

    Yet another alternative is to notice that the sum of the digits of N is the same as that of $\overline{N}$ and that since $\overline{N}$ is a multiple of 9, the sum of the digits $S(N) = S(\overline{N})$ must be a multiple of 9 as well. But 18 is the only multiple of 9 among the options.

15. A triangular grid of 11 points is given. How many triangles have all three vertices among the 11 points?



    (A) 140 **(B)** 141 **(C)** 142 **(D)** 150 **(E)** 165

    Solution: B. There are $\binom{11}{3} = 165$ three elements subsets of the grid, but $\binom{5}{3} + \binom{5}{3} + \binom{4}{3} = 10 + 10 + 4 = 24$ of these are colinear sets. Therefore there are $165 - 24 = 141$ sets of vertices.

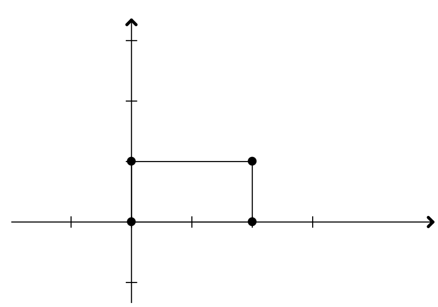
16. Let $f(x) = \frac{x-1}{x+1}$ and let $f^{(n)}(x)$ denote the n-fold composition of f with itself. That is, $f^1(x) = f(x)$ and $f^{(n)}(x) = f\left(f^{(n-1)}(x)\right)$. Which of the following is $f^{(2007)}(x)$?

    (A) $-\dfrac{1}{x}$ **(B)** $-\dfrac{x+1}{x-1}$ **(C)** $\dfrac{1}{x}$ **(D)** $\dfrac{1-x}{1+x}$ **(E)** $\dfrac{x-1}{x+1}$

    Solution: B. Computing $f^{(2)}(x), f^{(3)}(x), f^{(4)}(x)$ and $f^{(5)}(x)$, we see that $f^{(2)}(x) = -\frac{1}{x}, f^{(3)}(x) = -\frac{x+1}{x-1}, f^{(4)}(x) = x$ and $f^{(5)}(x) = f(x)$. Since the remainder r when 2007 is divided by 4 is 3 $(2007 = 4 \cdot 501 + 3)$, it follows $f^{(2007)}(x) = f^{(3)}(x) = -\frac{x+1}{x-1}$.

17. A point (x, y) is selected at random from the rectangular region shown. What is the probability that $x < y$?

    (A) 1/5 **(B)** 1/4 **(C)** 1/3 **(D)** 1/2 **(E)** 2/3



Solution: B. Exactly 1/4 of the rectangular region lies above the line $y = x$.

18. The number 240,240 can be expressed as a product of k consecutive integers. A possible value of k is

    (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

    Solution: B. Factoring the number yields $7, 11$, and 13 as factors. Since these are all prime, either they or one of their multiples must be among the k integers. Trying 13 itself, we write $10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 = 240{,}240$.

    To see that no other value besides $k = 5$ works, trying 26, the closest multiples of 11 are 22 and 33, but any product of consecutive integers including both 22 and 26 is too large, as is any such product containing both 26 and 33. The same reasoning holds for 39 and 52 (55 is the closest multiple of 11 and 53 is prime).

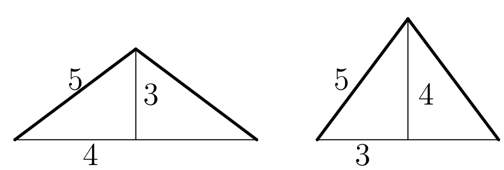
    Alternatively, "slowly" factor 240,240 into smaller numbers – $240,240 = 10 \cdot 24,024 = 10 \cdot 12 \cdot 2002 = 10 \cdot 12 \cdot 2 \cdot 1001 = 10 \cdot 12 \cdot 2 \cdot 11 \cdot 91 = 10 \cdot 12 \cdot 2 \cdot 11 \cdot 7 \cdot 13$. Simply multiply the "2" and "7" together and reorder to have $240,240 = 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14$. So $k = 5$ works.

19. Let A be the area of a triangle with sides $5, 5$, and 8, and let B denote the area of a triangle with sides $5, 5$, and 6. Which of the following is true.

    (A) $A < B < 12$ **(B)** $B < A < 12$ **(C)** $A = B$

    (D) $12 < A < B$ **(E)** $12 < B < A$

    Solution: C. Each triangle has area 12. To see this, construct for each triangle the altitude to the even length side and use the Pythagorean Theorem.



20. Suppose a, b and c are real numbers for which

$$\frac{a}{b} > 1 \text{ and } \frac{a}{c} < -1.$$

Which of the following must be correct?

(A) $a + b - c > 0$ **(B)** $a > b$ **(C)** $(a - c)(b - c) > 0$

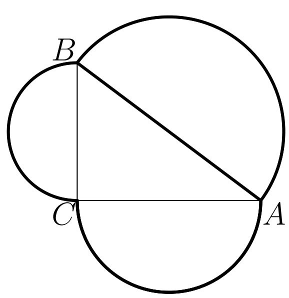
(D) $a - b + c > 0$ **(E)** $abc > 0$

Solution: C. The first inequality guarantees that a and b have the same sign, and the second that a and c do not. It follows that in either case, $a - c$ and $b - c$ have the same sign. Let $a = -2, b = -1$, and $c = 1$ to see that (A) and (B) fail. Let $a = 2, b = 1$, and $c = -1$ to see that (D) and (E) fail.

A lovely alternative is to note that the given inequalities are left unchanged by the replacement of $-a, -b$ and $-c$ for a, b, and c, so the correct option must also be left unchanged. Only option **C** has that property.

21. A right triangle ABC is given. Semicircles are constructed with the sides of the triangle as diameters, as shown below. Suppose the area of the largest semicircle is 36 and the area of the smallest one is 16. What is the area of the other one?

(A) 20 **(B)** 24 **(C)** 25 **(D)** 26 **(E)** 30



Solution: A. Let a, b and c denote the lengths of the three sides, $a = BC, b = AC$, and $c = AB$. Now $a^2 + b^2 = c^2$ since the triangle is right. The areas of the three semicircles are $(1/2)\pi(a/2)^2, (1/2)\pi(b/2)^2, (1/2)\pi(c/2)^2$. Therefore, we have $(1/2)\pi(a/2)^2 + (1/2)\pi(b/2)^2 = \pi(a^2 + b^2)/8 = (\pi/8)c^2$, so the sum of the areas of the two smaller semicircles is the area of the largest one. Thus the area of the middle one is $36 - 16 = 20$.

22. A standard deck of 52 cards contains 13 hearts. Twenty six cards have already been dealt, eight of which are hearts. If you are dealt 13 of the remaining cards, what is the probability that you will get exactly 2 of the remaining 5 hearts? (Round your answer.)

(A) 22% **(B)** 26% **(C)** 30% **(D)** 34% **(E)** 38%

Solution: D. There are $a = \binom{26}{13}$ ways to pick 13 of the 26 remaining cards. Since there are 5 hearts and 21 non-heart cards remaining, there are $b = \binom{5}{2}\binom{21}{11}$ ways to choose exactly 2 hearts. The probability b/a is $3527160/10400600 \approx 0.34$.

23. Find the value of the expression

$$S = 1! \cdot 3 - 2! \cdot 4 + 3! \cdot 5 - 4! \cdot 6 + \ldots - 2006! \cdot 2008 + 2007!$$

(A) -2007 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2007

Solution: D. Let us simplify S from the right end: $-2006! \cdot 2008 + 2007! = 2006!(-2008 + 2007) = -2006!$. Thus, $S = 1! \cdot 3 - 2! \cdot 4 + 3! \cdot 5 - 4! \cdot 6 + \ldots + 2005! \cdot 2007 - 2006!$. Next, $2005! \cdot 2007 - 2006! = 2005!(2007 - 2006) = 2005!$ Therefore, $S = 1! \cdot 3 - 2! \cdot 4 + 3! \cdot 5 - 4! \cdot 6 + \ldots - 2004! \cdot 2006 + 2005!$ Continuing simplification of S we finally obtain that $S = 1! = 1$.

Alternatively, for each n, $n! \cdot (n+2)$ can be rewritten as $n! \cdot (n+2) = n! + n!(n+1) = n! + (n+1)!$. Do this from the beginning to get $S = (1! + 2!) - (2! + 3!) + (3! + 4!) - \cdots + (2005! + 2006!) - (2006! + 2007!) + 2007! = 1! = 1$.

24. You bought a big cake for a party and expect 10 or 11 people to come. What is the minimal number of pieces (perhaps of different sizes) you need to divide the cake evenly if exactly 10 guests attend and also evenly among 11 guests?

(A) 11 **(B)** 20 **(C)** 30 **(D)** 55 **(E)** 110

Solution: B. In order to accommodate 11 guests, no single slice of cake can be larger that 1/11 of the cake. Therefore, if ten guests arrive, each guest must get at least two slices of cake. Consequently, we must divide the cake into at least 20 slices. Let us show that this number suffices. Divide the cake into 11 equal pieces. Then divide one of these into 10 equal pieces. (Each of these is 1/110 of the cake.) We now have a total of 20 pieces. If 10 guests are coming everybody gets one big and one small piece, so that $1/11 + 1/110 = 1/10$. If 11 guests arrive, then 10 of them get 1/11 each and the last one receives the 1/11 that was divided into 10 pieces.