UNC Charlotte 2012 Comprehensive

March 5, 2012

1. In the English alphabet of capital letters, there are 15 "stick" letters which contain no curved lines, and 11 "round" letters which contain at least some curved segment. How many different 3-letter sequences can be made of two different stick letters and one curved letter?

   Stick: A E F H I K L M N T V W X Y Z

   Round: B C D G J O P Q R S U

   (A) 2310 (B) 4620 (C) 6930 (D) 13860 (E) none of these

2. I have several quarters. If I divide my quarters into two unequal piles, then the difference of the squares of the number of coins in each pile is 24 times the difference of the number of coins in each pile. How much is my collection of quarters worth?

   (A) \$9 (B) \$8 (C) \$7 (D) \$6 (E) \$5

3. The area bounded by the graph of the function $y = |x|$ and by the line $y = c$ is 5. What is c equal to?

   (A) $\sqrt{5}$ (B) $2\sqrt{5}$ (C) 5 (D) 10 (E) 25

4. In the Carolina Cash 5 game, you have to select five different numbers from 1 to 39. You win some prize if two or more of your five chosen numbers match the five numbers drawn (order does not matter). What is the probability that you win some prize in a single drawing? Round your answer to the nearest percent.

   (A) (7%) (B) (8%) (C) (9%) (D) (11%) (E) (12%)

5. Let a and b be the two roots of the equation $x^2 + 3x - 3 = 0$. Evaluate the value $a^2 + b^2$.

   (A) 4 (B) 9 (C) 10 (D) 12 (E) 15

6. One sphere is inscribed in a cube, while the cube is also inscribed in another sphere. Find the ratio of the volumes of the larger sphere to the smaller sphere.

   (A) $\sqrt{3}$ (B) $2\sqrt{3}$ (C) $3\sqrt{3}$ (D) $3\sqrt{2}$ (E) none of these

7. Let ABC be an equilateral triangle with an inscribed circle of radius 1, find the length of AB.

   (A) $\sqrt{2}$ (B) $2\sqrt{2}$ (C) $\sqrt{3}$ (D) $2\sqrt{3}$ (E) $3\sqrt{3}$

8. Determine the sum of the y-coordinates of the three intersection points of $y = x^2 - x - 5$ and $y = -5/x$.

   (A) -5 (B) -3 (C) 0 (D) 3 (E) 5

9. Consider a five-digit integer of the form $a11bc$ (with $a \geq 1$). It is known that it is divisible by 45. How many such numbers exist?

   (A) 21 (B) 20 (C) 19 (D) 18 (E) 10.

10. Consider a general convex 7-gon with vertices A_1, A_2, ..., A_7 (marked in the order as they appear on the polygon). Connecting the first vertex A_1 with the third vertex A_3, the second vertex A_2 with the fourth vertex A_4, and so on, we get a "star" as shown in the figure below with the vertices A_1, A_2, ..., A_7. Find the sum of the angles $\angle A_1$ to $\angle A_7$.

    (A) 360° (B) 480° (C) 540° (D) 600° (E) 620°

11. After a ship wreck, a surviving mouse finds himself on an uninhabited island with one kilogram of cheese and no other food. On the 1st day, he eats 1/4 of the cheese, on the 2nd, 1/9 of the remaining cheese, on the 3rd, 1/16 of the

remainder, and so on. Let M be the total amount of cheese (measured in kilograms) consumed during the first 6 weeks. Which of the following statements is true?

(A) $0.4878 < M \leq 0.4881$ (B) $0.4881 < M \leq 0.4884$
(C) $0.4884 < M \leq 0.4887$ (D) $0.4887 < M \leq 0.4890$
(E) $0.4890 < M \leq 0.4893$

12. Two adjacent sides of the unit square and two sides of an equilateral triangle bisect each other. Find the area of the equilateral triangle.

(A) $\sqrt{3}/2$ (B) 1 (C) 0.8 (D) $\sqrt{2}$ (E) $\sqrt{2}/2$

13. Let C be a circle that intersects each of the circles $(x+2)^2 + y^2 = 2^2$, $(x-4)^2 + (y-2)^2 = 2^2$ and $(x-4)^2 + (y+2)^2 = 2^2$ in exactly one point and does not contain any of these circles inside it. If the radius r of C is of the form $r = p/q$ where p and q are relatively prime integers, what is $p+q$?

(A) 5 (B) 7 (C) 9 (D) 11 (E) 13

14. Three crazy painters painted the floor in three different colors. One painted 75% of the floor in red on Monday, the second painted 70% of the floor in green on Tuesday, and the third painted 65% of the floor in blue on Wednesday. At least what percent of the floor must be painted in all the three colors?

(A) 10% (B) 25% (C) 30% (D) 35% (E) 65%

15. After each mile on a highway from Charlotte (CLT) to the City of Mathematics Students (CMS) there is a two sided distance marker. One side of the marker shows the distance from CLT while the opposite side shows the distance to CMS. A curious student noticed that the sum of the digits from the two sides of each marker stays a constant 13. Find the distance d between the cities.

(A) $10 \le d < 20$ (B) $20 \le d < 30$ (C) $30 \le d < 40$ (D) $40 \le d < 50$
(E) $50 \le d < 60$

16. Consider the sum $1^2 + 2^2 + \cdots + 2012^2$. What is its last digit?

    (A) 6 (B) 5 (C) 3 (D) 1 (E) 0

17. Infinitely many empty boxes, each capable of holding six dots are lined up from right to left. Each minute a new dot appears in the rightmost box. Whenever six dots appear in the same box, they **fuse** together to form one dot in the next box to the left. How many dots are there after 2012 minutes? For example, after seven minutes we have just two dots, one in the rightmost box and one in the next box over.

    (A) 8 (B) 9 (C) 10 (D) 11 (E) 12

18. Consider the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21 \ldots$ where each term, after the first two, is the sum of the two previous terms. How many of the first 1000 terms are divisible by 3?

    (A) 200 (B) 250 (C) 299 (D) 300 (E) 333

19. The points A, B, C and D are the vertices of a unit square. How many squares (including $ABCD$ itself) in the same plane have two or more of these points as vertices?

    (A) 13 (B) 12 (C) 9 (D) 5 (E) 4

20. Three fair (six sided) dice are colored green, red and yellow respectively. Each die is rolled once. Let g, r and y be the resulting values of the green, red and yellow dice respectively. What is the probability that $g \le r \le y$?

    (A) $\frac{5}{27}$ (B) $\frac{6}{27}$ (C) $\frac{7}{27}$ (D) $\frac{8}{27}$ (E) $\frac{5}{18}$

21. For how many real numbers x is it true that $\lfloor \frac{x}{2} \rfloor + \lfloor \frac{2x}{3} \rfloor = x$ (where $\lfloor y \rfloor$ is the greatest integer n such that $n \le y$)?

    (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

22. A person is given a square-shaped pizza and a knife. He cuts the pizza with 40 straight cuts into as many pieces as possible. Denote the resulting number of pieces by N. Which of the following statements is true?

    (A) $831 \leq N \leq 840$ (B) $821 \leq N \leq 830$ (C) $811 \leq N \leq 820$
    (D) $801 \leq N \leq 810$ (E) $791 \leq N \leq 800$

23. Find the smallest five-digit integer such that the product of its digits is 2520. The sum of its digits is

    (A) 25 (B) 26 (C) 27 (D) 29 (E) 30

24. If a convex n-gon has no adjacent obtuse angles then n is at most

    (A) 4 (B) 5 (C) 6 (D) 7 (E) 8

25. You are in a large room with 50 ceiling lights (numbered from 1 to 50) that are changed from on to off or off to on by pulling a cord hanging from each light. Initially, all the lights are off. You begin by pulling the cord on every light (now they are all on). Then you pull the cord on light 2, 4, ..., 50. After you finish that, you pull the cord on light 3, 6, ..., 48. You repeat this with every fourth light, every fifth light, etc. until you pull the cord for every 50th light (only number 50, of course). How many lights are on at the end?

    (A) 1 (B) 2 (C) 3 (D) 5 (E) 7